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W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

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VOLUME XIV MARCH, 1923 NOS. 3 AND 4

A FIRST STUDY OF THE TIBETAN SKULL.

BY G. M. MORANT, M.Sc.,
Crewdson-Benington Student in Craniometry.

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1. *Introductory.*

THE present paper continues the craniometric studies which have already been carried out in the Biometric Laboratory and it is supplementary to Miss Tildesley's recent memoir on the Burmese skull. The writer hopes that it may be followed by papers dealing with material from the Himalayan border-lands and Eastern China. These peoples are of peculiar interest because the country which they inhabit is a borderland between Mongolian and Indian races, and in prehistoric and early historic times was probably traversed by nomadic tribes in their wanderings. The general tendency of anthropologists now-a-days is to lay smaller stress on the value of local races and to seek for fundamental human types which may be mingled or hybridised in local races. All the races at present inhabiting the country between India and China and the Chinese themselves are supposed to constitute such a type which is usually termed "mongolian" and is characterised by a brachycephalic skull and a broad, flattened face. But in the Tibetan province of Khams there is a dolichocephalic people with no signs of flattening of the facial bones. They have heavy and capacious crania with strong muscular development and other characters which clearly differentiate them from Chinese, Malayan and Burmese skulls. No relationship can be suggested with the dolichocephalic peoples of India for the crania of the latter are feeble and under-sized. Comparison is more possible with Moriori and Fuegian crania than with those of any of their immediate neighbours east and west of which we have knowledge. That the cranium of the Khams Tibetans

should be so widely differentiated from that of other Orientals is a fact of great suggestiveness and it shows how important the study of local races may be.

In India and Tibet there are two races which are obviously unrelated, and yet the cephalic indices of the two are almost identical. This is but one illustration of the inadequacy of that criterion when used as a measure of racial affinity. The writer has been led to the conclusion that the cephalic index alone is quite incapable of discriminating between fundamental types or of distinguishing relationships between races which are known to be allied. Furthermore, no single character which has yet been suggested can fulfil either of these purposes and it is extremely unlikely that one will ever be found.

The object which the writer of this paper has kept in view has been the means of distinguishing biometrically between various types of man as far as such are represented in greater or less dominance in local races. The two methods thus far adopted by the biometric craniologists have been a comparison of type contours and the admittedly temporary "Coefficient of Racial Likeness." This latter method furnishes a general measure of racial resemblance which will give weight to all craniometric characters, or at least to a considerable number of well chosen characters, and it is intended to supplant the old methods of comparison between single characters such as the cephalic index or the nasal index. There is no great difficulty theoretically in devising a reasonable measure of racial resemblance provided we had an adequate knowledge of:

(i) the means of some 30 to 40 characters in 50 to 100 local races;

(ii) the standard deviations and correlations of these means among themselves—i.e. inter-racial variations and inter-racial correlations.

Unfortunately the material for (i) and therefore (ii) is very inadequate at present.

Therefore we are thrown back for the time being on the "Coefficient of Racial Likeness" which Professor Karl Pearson has suggested. This is a measure of whether any two races can be considered samples of the same population. In actual use it depends on the assumptions:

(i) that there is intra-racially no *high* correlation between the characters selected for comparison;

(ii) that failing intra-racial variability being adequately known—especially for short series—it will be sufficient to assume equal variabilities for all races, and to use for these variabilities those obtained for a really long series of a single race.

Such a method cannot be final or completely satisfying to the statistically trained mind, but it may well serve, as it was originally intended to do, as a stop-gap, until sufficient material has accumulated for its replacement by an ampler procedure.

But the assumptions on which it is based are not so contrary to experience as to lead us to discard what aid we can get from the "Coefficient of Racial Likeness." There is in fact at the present time nothing to supply its place. Our aim should rather be to determine how far it leads us to the same conclusions as we can draw from type contours, and it has this great advantage that it enables us by the very process of its deduction to ascertain in which characters two races significantly

diverge from one another. It may be, as the comparisons already made partly suggest, that there are some characters which differ between race and race far more than others do, and if a selection of the characters which fluctuate most could be made it might be possible to neglect the more constant ones.

The other method of estimating the significance of racial differences is that based on a comparison of type contours. The three type or mean racial contours are constructed from the mean lengths of the individual contours of a racial series and on these three there are no adequate representations of some important regions of the skull. For this reason, and also because the conclusions to be drawn from drawings necessarily lack the precision of arithmetical analysis, the contours can never rival in importance the system of direct measurements. Nevertheless graphical methods of comparison appeal to many craniologists who suspect all statistical deductions and one of the chief objects of this paper has been to study how far they are helpful in the comparison of human types. But no very definite conclusions could be arrived at because at present we have type contours for only a few races and there has been a rapid evolution in the technique of representation so that considerably more of the skull is now portrayed than was shown on the earlier drawings. As the problem of human types now stands, we cannot assert that geographical proximity naturally connotes racial affinity. Cultures have spread from one end of the world to the other and the bearers of those cultures have undoubtedly in many cases gone with them. The type contours of palaeolithic man differ widely from those of modern man; those of Khams Tibetans and Moriori show some resemblance. Are we to argue in one case that difference associated with a wide temporal interval is significant, but on the other resemblance with a wide geographical interval can mean nothing? The present paper only touches the fringe of matters when it endeavours to contrast results obtained from the type contours with those suggested by other methods. But it is already clear that no classification of *normae verticales* can be a reliable guide to racial relationships. The *norma lateralis* is a more important aspect, but if considered alone it may lead to quite erroneous conclusions. Contours are also of importance because certain measures of curvature can be derived from them and these could not be easily determined directly.

The last object of this paper has been to develop the biometric methods of studying the human mandible. The few Tibetan mandibles which were available did not themselves justify this purpose, but by dealing with even so few it was possible to discover the difficulties of technique which practice alone can teach. A review of the definitions of mandibular "standard planes" and "points" and the methods of measurement provided in some text-books of anthropometry suggested that they had been drawn up before they were applied in practice, and accordingly after experience it became necessary to modify them considerably. The system of measurements has to be a detailed one if comparison is to be made with the mere fragments of jaws through which alone certain races of fossil man (and apes) are known to us. The type contours of the mandibles of the two Tibetan groups will appear in a subsequent memoir.

The subject of the present paper is a collection of Tibetan skulls made by Colonel L. A. Waddell in the south-western and eastern districts of Tibet and presented by him to the Royal College of Surgeons. With the kind permission of Sir Arthur Keith I have been able to measure these crania.

There are 32 skulls in the collection which is divided into two groups. The first group (Type A) comprises 17 skulls which were obtained from graves in Sikkim and neighbouring regions of Tibet proper; the remaining 15 skulls (Type B) were picked up on a battle-field in the Lhasa district and were believed to be those of native soldiers from the eastern province of Khams. With the exception of one skull of the B type, the parietal bones of which had been badly damaged by a bullet, they were all in a very good state of preservation. There are 16 mandibles of the A series and 12 of the B, but 5 of the latter did not correspond with any of the skulls. All the skulls of the B series were of a heavy and decidedly masculine adult type, but a few of the A had not reached maturity and the sex of several could not be positively determined.

Such material, though no very reliable means can be reduced from it, is of peculiar interest because of the rarity of Tibetan crania: we are only aware of the existence of five specimens from the interior regions of the country in any other European museum*. In spite of the meagreness of our material we ought to be able to discover any striking resemblances or differences there may be between the skulls of the Tibetans and those of neighbouring peoples.

2. *Measurements and Methods of Measurement.*

The following measurements are with certain exceptions those of the *Frankfurter Verständigung* taken in the manner described by C. D. Fawcett† together with those added by subsequent workers in this laboratory.

F = Flower's Ophryo-occipital length. L' = glabellar projective horizontal length. L = maximum length from glabella to occiput in the median plane. B = maximum horizontal breadth on parietal bones. B' = least forehead breadth from one temporal crest to the other. H' = basio-bregmatic height. H = height measured on craniophor, from basion to the point vertically above it (generally a little behind the bregma). OH = auricular height, as measured on the craniophor, that is, the height of the skull above that part of the auricular orifice which rests on the top edge of the ear-plugs‡. LB = length from basion to nasion. Q = transverse arc perpendicular to the horizontal plane, passing through the "apex," and terminating on both sides at the top of the ear-rods, this measurement being taken when the skull is on the craniophor. Q' = a similar measurement taken off the craniophor

* Dr W. L. H. Duckworth has kindly furnished me with a description of two Tibetan skulls in the Anatomical Museum at Cambridge. These were obtained in the southern province of Dokthol and appear to be of the Tibetan A type. There are three skulls from the interior of Tibet and many more from the south-west border-land of the country in the Hodgson collection at the British Museum (Natural History).

† *Biometrika*, Vol. I. pp. 412—419.

‡ The accuracy and significance of this measurement, which has become of peculiar importance since cranial contours have been drawn, will be discussed later.

and terminating at the "auricular points*." S = sagittal arc from nasion to opisthion. S_1 = arc from nasion to bregma. S_2 = arc from bregma to lambda. S_3 = arc from lambda to opisthion. S_3' = chord from lambda to opisthion. U = horizontal circumference measured directly above the superciliary ridges and round the most projecting sagittal part of the occiput, i.e. the most projecting part between the inion and lambda. PH = premaxillary height, measured from alveolar point to tip of anterior nasal spine, if such exists, and, where it does not, to the sagittal point of the lower edge of the pyriform aperture. $G'H$ = upper face height from nasion to alveolar point. GB = face breadth from the lower end of one zygomatic-maxillary suture to that of the other, where the suture crosses the lower front rim of the cheek-bone. J = zygomatic breadth, from the most lateral point of one zygomatic arch to that of the other. NH, R and L = nasal height from nasion to lowest edge, right and left, of pyriform aperture. NB = nasal breadth, greatest breadth of pyriform aperture, wherever it may be. DS = dacryal subtense, being the shortest subtense from the bridge of the nose to the dacryal chord, measured with Mérejkowsky's simometer. DC = chord from dacryon to dacryon. DA = shortest arc over the bridge of the nose from dacryon to dacryon, measured with the tape. SS = simotic subtense, the shortest subtense from the nasal bridge to the simotic chord, as measured by the simometer. SC = minimum chord between the two naso-maxillary sutures. O_1 = greatest breadth of orbit, R and L, from side to side, using Fawcett's curvature method† to determine the margin on the nasal side. O_1' = breadth of orbit from dacryon. This measurement is only taken because it is frequently given by other workers for other races. O_2 = greatest height of orbit, R and L, taken perpendicular to O_1. G_1 = length of palate, from the point of the *spina nasalis posterior* to an imaginary line tangential to the inner rims of the alveoli of the middle incisors. G_1' = similar measurement to G_1 but taken from base of spine. G_2 = breadth of palate between inner alveolar walls at second molars. GL = profile length from basion to alveolar point. fml = length of *foramen magnum* from basion to opisthion. fmb = greatest breadth of *foramen magnum*. $P\angle$ = profile angle, or angle made by the line from the nasion to alveolar point with the horizontal, found by means of Ranke's Goniometer when the skull is in the horizontal position on the craniophor, as described by Fawcett (*Biometrika*, Vol. I. p. 414). Various indices are calculated from these direct measurements: occipital (*Oc. I*), cephalic (100 B/L', 100 H/L, 100 H/L', 100 B/H), facial (100 $G'H/GB$), nasal (100 NB/NH), orbital (100 O_1R/O_2R, 100 O_1L/O_2L), palatal (100 G_2/G_1), foraminal (100 fmb/fml), dacryal (100 DS/DC), simotic (100 SS/SC). The angles ($N\angle$, $A\angle$, $B\angle$) of the fundamental triangle whose angles are the nasion, alveolar point and basion, were calculated from the measured lengths LB, GL, $G'H$ with the aid of Pearson's Trigonometer in the manner described by Fawcett‡. θ_1, the basio-nasal horizontal angle, is obtained by subtracting $N\angle$ from the supplement of $P\angle$; θ_2, the basio-alveolar horizontal angle, by subtracting $A\angle$ from $P\angle$.

* These are defined on p. 201. † See *Biometrika*, Vol. I. p. 430 and Vol. VIII. pp. 311, 312.

‡ *Biometrika*, Vol. I. p. 418.

Of the direct measurements above L', H', H, OH, Q and $P\angle$ are taken when the skull is in the Frankfurt horizontal position on the craniophor; Q, Q', S, S_1, S_2, S_3 and U are made with the steel tape; F, L, B, B', LB, S_5' and J with Flower's callipers; DS and SS with the simometer and the remainder with small callipers. The small arc from dacryon to dacryon (DA) was not measured with the steel tape but indirectly by marking its extremities on a small slip of paper stretched over the bridge of the nose.

To test the supposed correlation between intelligence or size of skull and height of palate*, some means of measuring the latter is required. The sagittal contours will give the height alone, but by means of an instrument known as the "Uraniscometer," designed by Professor Karl Pearson and shortly to be described by him, it is possible to measure directly both the height and breadth of the palate. The breadth (EB) was taken between the inner alveolar walls at the second molars and the height (EH) from the line joining the extremities of the chord (EB) and in the transverse plane through them. Thus EB is precisely the same measurement as G_2 made with the small callipers and it was only repeated to test the accordance of the two methods of determining it. The greatest difference between them was 0·5 mm., so G_2 need not be taken in future. $100\,EH/EB$ provides a second palatal index.

The only other direct measurements taken were those of the weight and capacity of the skull. The latter was determined by tight packing with mustard-seed, weighing the contents and estimating the capacity indirectly by filling a prepared skull first with seed and then with water. This method is precisely that used and fully described by Macdonell†.

After some preliminary practice a skull (T 27) was packed as tightly as possible with mustard-seed which was then weighed. Repeating this operation 6 times gave the weight of the contents in grams:

(1) 1332·2, (2) 1345·4, (3) 1349·9, (4) 1351·6, (5) 1353·8, (6) 1350·3.

With the exception of the first these results appeared to agree quite as well as we could expect. The weight of seed which each skull would hold was then determined. After an interval of a week this operation was repeated and a fairly uniform increase in weight on the first readings was found, though it never exceeded 15 grams. The skull T 27 held seed weighing 1357·2 grams, an increase of 3·4 on the maximum for the first occasion. There had been no practice in the intervening period and the conditions and method of packing appeared to be the same on both trials. Though showing how large the "personal equation" might be, the difference between the two determinations would be of no consequence if we could feel confident that all the skulls in the second case had been packed with the same degree of tightness. There was good reason for supposing that such had been the case, for the weight of seed held by the "crâne étalon" δ had been found once on every day of the period over which the second weighings extended, giving in grams:

(1) 1062·2, (2) 1060·2, (3) 1056·9, (4) 1061·7, (5) 1058·7.

* See *Biometrika*, Vol. XIII. p. 231. † *Biometrika*, Vol. III. pp. 203—206.

These show that the method of procedure was as nearly the same for all the skulls as we could reasonably hope it would be, and accordingly the second series of weights was used to determine the capacities. Filling the *crâne étalon* with water, of which the volume was afterwards found by pouring it into a measuring cylinder, gave in cm.^{3}:

(1) 1350, (2) 1360, (3) 1365.

Much greater care was taken in filling on the last occasion than on the previous ones and the increase is very significant. Accepting the last volume and taking the weight of seed held by the skull to be 1062·2 grams, gives a weight of 778·2 grams corresponding to a volume of 1000 cm.^{3}. Repeating this process with the *crâne étalon* γ gave a weight of 777·2 grams for the same volume. Using the standard skull δ Macdonell found that 1000 cm.^{3} contained 764·6* grams of seed and Tildesley's result for the same skull was 779·9 grams. To test the accuracy of the method, the seed that had been packed into the prepared skull δ was turned into a measuring cylinder, as well as being weighed, and pressed down as tightly as possible. The mean capacity found in this way was 1349 cm.^{3}, which shows that the seed could be compressed rather more easily in the glass vessel than in the skull. The difference in the results given by the two methods is small, but the former is almost certainly the more reliable. The final capacities were calculated on the supposition that 1000 grams of seed occupied a volume of 1000/778·2 cm.^{3}.

The only other direct measurement taken was that of the weight of the skull. No attempt was made to make allowance for lost teeth or missing portions of bones.

The occipital index (*Oc. I.*), which is defined to be $100\,\frac{S_2}{S_3'}\sqrt{\frac{S_3}{24\,(S_3-S_3')}}$, was found with the aid of Miss Tildesley's table of the function, given in the appendix to her Burmese paper.

A few remarks must be made on the methods of measurement used by biometric craniologists. Special care has to be taken when compiling comparative material as a few of the letters denoting measurements have not been used by different workers with precisely the same significance.

H and H'. The *H'* defined above is the *H* of Macdonell, Benington and Thomson.

Q and Q'. The *Q* of Macdonell and Benington was taken over the bregma, instead of in the vertical plane. Benington's *Q'* is our *Q*. We have used the same *Q* as Thomson, but her *Q'* was taken from the upper margin of one auricular passage to that of the other.

O_1R and O_1L. We have followed Fawcett, Thomson and Tildesley in using the "curvature" method to determine the width of the orbit. Macdonell and Benington used the "geodesic†."

NH. This measurement is frequently lacking in precision owing to the difficulty of determining exactly where the lowest point of the edge of the pyriform aperture

* The seed used had been kept for a number of years after Macdonell determined capacities with it; it may have become more shrunk during that time, and the *crânes étalons* had been repacked with cement.

† See *Biometrika*, Vol. I. p. 431.

is. Thomson measured to the lowest point on the left side and Fawcett, Macdonell and Benington to the lowest point whether right or left. Tildesley was the first to measure both sides and we have followed her in this. The Tibetan crania were apparently more symmetrical in this respect than the Burmese.

DA and DC. These were first measured by Benington and, before Mérejkowsky's simometer was used, the subtense *DS* was obtained from them by calculation.

OH. The error detected by Tildesley in the vertical scale of the Ranke's craniophor used in the Biometric Laboratory was confirmed to be 2·5 mm. and that amount was accordingly subtracted from all the scale readings of *OH*. This correction was made to the means of previous workers before using them for comparative purposes*.

Sex and Age. The skulls were examined by Professor Pearson with the object of sexing them. The Tibetan *B* skulls were more bleached and could be readily distinguished from the others. They were undoubtedly all adult males of quite a massive and heavy type and with well marked glabellas and inions. The Type A skulls were smaller and lighter and could not be sexed so easily. The majority of these were undoubtedly male, but three of them were young, with third molars appearing but not fully erupted, and the sex of one or two was doubtful. A few were certainly female, but the series was such a short one, comprising only 17 skulls, that it was thought best to pool them all to obtain racial means. This would have the effect of making the mean direct measurements slightly less than for an entirely male population, but the indicial and angular measurements would be less definitely affected.

Mandibles. The mandibular measurements made are described in Appendix I to this paper.

3. *On the Difficulties attached to the Location of certain Cranial "Points."*

Inion. No direct measurements terminate in this point but it is marked on the sagittal contours. In the cases where the inionic prominence was clearly marked the "point" was taken to be the meeting place of the *lineae nuchae supremae* and this was found to be very close to the common horizontal tangent of the *lineae nuchae superiores*†: see Plate XIV. The intersection of that line with the median plane determined the inion in less clearly defined cases. But often its position could only be vaguely conjectured.

Basion and Opisthion. These points were not marked on the skull, but the divergencies between the measurements terminating in them, taken on two separate occasions, were no larger than the average. The differences which were observed between the mean foraminal lengths deduced from the sagittal contours and direct measurements were more significant. They would probably have been less if the points had been marked on each skull before the contours were drawn.

* A table of these corrected means is given in *Biometrika*, Vol. XIII. p. 217.

† When determining the position of the inion the writer was especially guided by a discussion of the subject in *Biometrika*, Vol. XIII. p. 184.

Auricular Points. The difficulties of defining clearly the positions of the auricular points have been discussed at considerable length by many craniometricians. It seems best to accept and use in practice two definitions which will give very close but not always coincident points.

The craniophor auricular points are defined to be those at which the skull, when in the Frankfurt horizontal plane, touches the ear-rods of the craniophor. The ear-rods were only inserted a short way into the auricular passages; almost as short a distance as was consistent with stability. The *OH* height is taken from the top of the rods and the divergencies between the two measurements for each skull found on separate occasions had a maximum of 2 mm., but they were only more than 1 mm. in three cases. These craniophor points are the ones shown on the sagittal and transverse vertical contours and though they had not been marked on the skulls, there seemed to be no difficulty in roughly locating them again. A comparison of the three mean *OH* measurements—one by direct measurement and the other two from the contours—showed a maximum difference of 1·0 mm. (see p. 251 below). This is larger than one would like, but it could probably be made less by marking the points on the individual skull while it is still on the craniophor and before drawing the contours.

The auricular points were defined by Fawcett to be "the highest points of the upper rims of the auricular passages*," and these will not always coincide with the points of contact of the ear-rods. It was found that there was a thin lip of bone on nearly every skull, terminating posteriorly in a well marked notch, and coinciding almost exactly with any line that could be supposed to mark the upper rim of the auricular passage. The point of intersection of that lip of bone with the plane bisecting the orifice transversely was taken to be the "auricular point by definition" at which the arc Q' terminates.

4. "*Remarks*†" *on Individual Crania.*

The "remarks" made on out-standing cranial anomalies are similar to those of previous biometric craniometricians. The number of skulls (32) with which we were dealing was, for statistical purposes, so small that no reliable conclusions at all could be drawn as to the racial significance of the proportions of the total population having or lacking any particular character.

The following characteristics and anomalies were looked for and, if no mention of one of these anomalies has been made, it is to be definitely understood that it did not exist in the skull.

Age and signs of age. Where no comment is made the skull is adult with sutures joined but not obliterated; basilar synchondrosis noted; also any teeth in process of coming through or lost during life and consequent absorption of alveolus; falling-in or thinning of calvaria, due to old age, looked for.

Teeth. Undeveloped or imperfectly developed third molars; extreme attrition of teeth and any irregular formations round palate; diseased teeth. The teeth of

* *Biometrika*, Vol. I. p. 413.

† These are given, together with the individual measurements, in Appendix II to this paper.

the majority of the Tibetan skulls, and particularly those of the B type, were in a very good state of preservation and hardly any showed signs of disease or wear.

Palate. Each was examined for the existence of bony ridges across the palatine grooves leading from the pterygo-palatine canals*. When such a one is thrown over the inner groove on either side, it is described as an inner palate bridge, left or right; over the outer groove as an outer palate bridge. One inner palate bridge was found and there were no cases of an outer one. But on several of the palates there were small spikes of bone on both sides of the canals which might well have been the broken extremities of what was originally a complete bridge. The palate was also examined for palatine torus, a thickening of the bone on both sides of the median suture.

Precondyles. The existence of a pair of small, distinct precondyles was noted in the case of two skulls and there was also one in which the precondyles had become completely fused: see Plates XIII and XIV.

Tympanic perforation. Perforation of the tympanic plate was recorded in all cases.

Base of the Pyriform Aperture. This feature has been indicated by the letters P.B., and the description has reference first to that part of the floor of the nasal cavity which lies immediately behind the edge of the aperture, and secondly to the edge itself. The floor is described as "flat"—in which case it may be assumed to be approximately horizontal as well—or "sloping upwards and outwards," or "downwards and outwards"; the edge itself is described as sharp, blunt or rounded, and if it is double with a groove between its two parts, that is also mentioned. In some cases the floor of the cavity is so rounded as to form part of an almost unbroken curve with the rounded lip: such a case is described shortly as P.B. rounded, or V. rounded. The following frequencies were found for the Tibetan skulls:

Type of Edge	Type A	Type B
Sharp ...	12	7
Blunt ...	4	7
Rounded ...	1	1

There were three skulls having a double edge to the pyriform aperture.

Asymmetry. Each skull is marked SR or SL according as the Sylvian depression is greater on the right or left side, and, in the same way, JR or JL has reference to the greater size of the right or left jugular foramen. The frequencies of these characters were:

	Sylvian depression	Jugular foramen
Greater on right side	18	26
Equal	13	2
Greater on left side	1	4

The balance in favour of the right side shown here is similar to that which was observed in the case of Burmese crania†. All cases of asymmetry of other parts of the skull are noted; only 4 skulls were at all markedly asymmetrical.

* See Le Double, *Variations des Os de la Face*, p. 266.

† M. L Tildesley, "A First Study of the Burmese Skull." *Biometrika*, Vol. XIII. pp. 253, 255.

Malar bones. There were no cases of a complete horizontal suture across the malar bone, although in several there were slight traces of it.

Metopism. Six of the 32 Tibetan skulls had a persistent frontal suture: this is a very unusually high percentage.

Ossicles in the sutures. All ossicles of any considerable size have been noted. If one side of the lambdoid or coronal suture is not indicated, it is to be understood that where wormians are mentioned they were present on both sides. One fairly marked case of wormians of the lambdoid suture is figured Plate XII.

Conformation at the pterion. All cases of epipteric bones have been mentioned; also instances of the thrusting of a process of the squama temporalis between the parietal and sphenoid bones to join the frontal.

Interparietal bones.* There was one case of a simple interparietal bone, two of the bipartite (rather different from that figured by Benington, *Biometrika*, Vol. VIII. p. 328) and no cases of the tripartite interparietal: see Plates XI and XII.

Other features. All cases of flattening of the obelion have been recorded together with any other singularities worthy of comment.

5. *Comparative Material.*

The greater part of Tibet is inhabited by seminomadic pastoral tribes.

"With the exception of the extreme northern and northeastern portions.....of Tibet, the population belongs essentially to one race, and the purest representatives of this stock are to be found among the pastoral tribes, or Drupa, which, whether found around the Kokonor, in eastern, western, or central Tibet, offer a uniform type which may be called the *Drupa type*†."

These people are said to be short and brachycephalic with high cheek bones, "but not so high as with the Mongols," noses usually narrow and strong but irregular teeth. This supposed primitive and pure type only persists among the nomadic tribes, while a hybrid one occupies the places of permanent habitation.

"It (the latter) is a mixed race, becoming more Chinese as one goes toward China, or more Indian (Nepalese or Kashmiri) as one travels southward or westward. The reason of the very pronounced departure of this portion of the Tibetan population from its original type is easily accounted for in the custom of foreign traders, soldiers, pilgrims, or officials inhabiting the country, of never bringing their wives into Tibet, but taking native concubines, a custom common in most parts of Asia. In as small a population as that of Tibet, which does not probably exceed 3,000,000, where the principal centres of population are and have been inhabited by comparatively large numbers of foreigners at least, this profound alteration of the primitive type is easily accounted for in this manner‡."

Our crania of Type A are, in all probability, those of such people, for they come from the line of the trade route from Bengal through Sikkim to Lhasa along

* See *Biometrika*, Vol. III. p. 220.

† W. W. Rockhill, "Notes on the Ethnology of Tibet" (1895), p. 673. From the *Report of the U.S. National Museum for* 1893.

‡ Rockhill, *op. cit.* p. 674.

which all but the nomadic tribes live. Indian traders do not, as a rule, themselves proceed beyond the confines of Tibet; their merchandise is carried to Lhasa by the Nepalese and Bhotanese. It is to these peoples that the supposed hybrid natives are probably racially akin, but unfortunately there are at present no data which can be used to test such a theory. It must not be forgotten that a few of the Type A crania come from Lhasa where the floating population is predominantly Chinese*.

Of the inhabitants of Khams, as the eastern portion of Tibet is called, W. W. Rockhill writes†, "there is absolutely nothing Mongol about this people, who are good representatives of old Tibetan civilisation." There are few Chinese traders, soldiers or officials in most parts of the country, but their number increases rapidly as the thickly populated boundary of western China is approached. We may hope then that the Type B crania are representative of the aboriginal inhabitants of the country and we should expect them to differ significantly from the hybrid Type A.

For comparative purposes we are able to use the table of means of several Asiatic races compiled by Tildesley from various scattered sources‡. This material is scanty and in some cases of very doubtful homogeneity, but together with the three Burmese series with which it has already been compared, it is, at present, all that is available and has been measured by methods sufficiently like ours to be comparable. The table of means is based on measurements given for the following races. We need only consider male skulls.

Malayans. These means are based on 78 skulls contained in a single collection.

Chinese. Measurements were collected from various sources and, although the part from which some of the skulls had been obtained was not known, the means may be taken to represent most nearly the South Chinese type.

Hindus. Only a portion of this racially heterogeneous people is represented. The material considered relates to Bengal and north-east India, all Dravidian tribes being excluded.

Dravidians. These are of the Maravar tribe from the Madras Presidency.

Burmese. Finally, there are the three Burmese series measured by Miss Tildesley and these were compared with the above racial material she had collected. The A series was of Burmans proper, the B of supposed hybrids and the skulls of the C series were thought to be of Karen origin.

Aino. These means, which were reduced in the Biometric Laboratory in 1900 from Koganei's tables, are quoted in *Biometrika*, Vol. I. p. 426.

The standard deviation of the cephalic index (B/L) may be conveniently taken to be an approximate measure of the homogeneity of a series of skulls, but it must not be forgotten that there is some evidence to show that advanced races are more variable than primitive ones.

* See L. A. Waddell, *Lhasa and its Mysteries* (1905), pp. 344—346.

† *The Land of the Lamas* (1891), p. 188 (footnote).

‡ *Biometrika*, Vol. XIII. p. 239.

Race	S.D. of B/L	Number of Crania
Aino*	2·41 ± ·18	88
Naqada*	2·80 ± ·17	130
Whitechapel English† ...	3·26 ± ·20	131
Tibetan A (Hybrids?) ...	3·87 ± ·66	17
Burmese A (Burmans) ...	4·30 ± ·46	44
Chinese	4·52 ± ·29	73
Tibetan B (Khams) ...	5·00 ± ·94	15
Malayans	5·13 ± ·28	77

It is curious, though no stress could be laid on the point without more complete evidence, that the variabilities in this index of the Oriental races, with the exception of the Aino, are significantly greater than those given for Western peoples‡. The probable errors of the standard deviations of the two Tibetan types are, of course, very large, but we may reasonably conclude that the collections are as homogeneous as those of the other Asiatic races with which they are to be compared. Table I (p. 206) gives a full list of mean measurements for the two Tibetan series and the other Asiatic ones with which comparison is to be made.

6. *The Coefficient of Racial Likeness.*

If an adequate comparison of the mean characters of two races is to be made, we require a knowledge of inter-racial standard deviations and correlations (i.e. of those of different races). The comparative material at present available is too meagre to provide even an approximation to these. Professor Karl Pearson has suggested that a single coefficient, known as the Coefficient of Racial Likeness (C. L.), which takes into account a large number of mean characters and requires intra-racial standard deviations (i.e. those of individuals all of one race) only, may be used provisionally.

Let m characters be measured and let the sth character in the first race have M_s for mean and σ_s for standard deviation, these two constants being based on n_s individual measurements. Let the corresponding characters for the second race be σ_s' and n_s'.

Then, if the two races were really samples of the same population, the following expression would be sensibly zero:

$$\frac{1}{m}\left\{\frac{S(M_s - M_s')^2}{\frac{\sigma_s^2}{n_s} + \frac{\sigma_s'^2}{n_s'}}\right\} - 1,$$

and the probable error of its deviation from zero will be $\cdot 67449/\sqrt{2m}$.

The value of this expression computed from a number of mean characters of two races is the Coefficient of Racial Likeness between them and it is thus a measure of the probability of the two being random samples from the same population. It is not a true measure of absolute divergence, and must not for a moment be considered as such, but nevertheless we shall speak of it, for convenience, as if it were an absolute measure of racial affinity. When it is said that

* Quoted by Fawcett, *Biometrika*, Vol. I. p. 440. † Macdonell, *Biometrika*, Vol. III. p. 224.

‡ The variabilities of the nasal and orbital indices of the Oriental races are also significantly greater than those of the Egyptians and Whitechapel English.

TABLE I. *Comparative Table of Means* (all Male). Asiatic Races.*

Character	Tibetan A	Tibetan B	Burmese A	Burmese B	Burmese C	Malayan	Chinese	Maravar	Hindu	Aino
Capacity	1433·0 (17)	1537·7 (14)	1406·9 (27)	1415·0 (4)	1442·2 (5)	[1424·4 (76)]	[1467·6 (46)]	[1289·7 (17)]	[1319·9 (34)]	[1462 (76)
W. in grs.	525·4 (17)	673·1 (15)	656·8 (38)	595·6 (7)	578·0 (6)	—	—	—	—	—
F	174·4 (17)	184·5 (14)	172·4 (44)	172·4 (8)	175·5 (8)	—	179·6 (16)	175·6 (17)	175·9 (45)	—
L′	175·2 (17)	185·7 (14)	174·2 (44)	176·7 (7)	177·2 (8)	—	179·2 (4)	—	173·2 (9)	—
L	174·8 (17)	185·5 (14)	173·5 (44)	173·8 (8)	176·7 (8)	174·7 (78)	177·1 (84)	175·6 (21)	175·4 (33)	185·8 (88)
B	139·4 (17)	139·4 (14)	143·7 (45)	141·1 (7)	140 4 (8)	142·2 (77)	139·5 (102)	131·4 (38)	132·3 (69)	141·2 (88)
B′	92·6 (17)	94·3 (15)	94·3 (44)	89·7 (8)	90·4 (8)	93·4 (77)	93·9 (49)	93·2 (21)	92·4 (10)	96·2 (88)
H	132·0 (17)	134·8 (15)	136·8 (43)	136·6 (7)	140·1 (8)	136·2 (76)	136·6 (37)	132·5 (38)	132·1 (43)	—
H′	131·2 (17)	134·1 (15)	136·0 (43)	134·7 (7)	139·1 (8)	137·4 (76)	136·9 (69)	—	131·5 (10)	139·5 (88)
OH	113·2 (17)	115·5 (15)	117·7 (44)	116·7 (7)	116·9 (8)	118·1 (77)	119·2 (38)	—	111·4 (10)	119·3 (88)
LB	95·7 (17)	99·2 (15)	98·5 (43)	98·8 (8)	100·5 (7)	99·5 (76)	99·1 (66)	98·8 (38)	99·2 (44)	105·4 (88)
Q	310·5 (17)	312·1 (15)	323·7 (44)	319·2 (7)	317.8 (8)	—	—	—	—	328·5 (84)
Q′	310·9 (17)	312·9 (14)	325·8 (43)	319·5 (7)	319·4 (8)	319·0 (77)	321·2 (30)	—	302·9 (10)	—
S	360·9 (17)	378·6 (14)	363·7 (44)	367·0 (7)	367·4 (8)	365·7 (73)	370·3 (78)	—	363·5 (33)	372·8 (77)
S_1	125·8 (17)	129·3 (15)	128·6 (44)	124·9 (7)	127·9 (8)	—	126·6 (57)	127·0 (21)	127·1 (33)	—
S_2	122·6 (17)	126·1 (14)	124·0 (45)	124·4 (7)	122·5 (8)	—	126·0 (56)	—	126·5 (33)	—
S_3	112·5 (17)	122·7 (15)	111·2 (44)	117·7 (7)	117·0 (8)	—	116·1 (56)	—	110·1 (33)	—
S_3'	94·4 (17)	100·3 (15)	94·7 (44)	98·1 (7)	99·7 (8)	98·0 (73)	98·7 (19)	—	92·3 (9)	—
U	503·6 (17)	525 6 (14)	505·7 (44)	497·7 (7)	503·1 (8)	505·7 (77)	508·5 (82)	488·7 (38)	493·5 (69)	522·5 (88)
PH	18·9 (15)	22·0 (15)	19·9 (38)	17·2 (7)	20·1 (7)	—	—	—	—	—
G′H	69·4 (15)	76·5 (15)	71·4 (39)	68·2 (8)	74·5 (7)	70·1 (73)	71·2 (49)	—	63·8 (9)	69·8 (73)
GB	98·7 (17)	100·7 (15)	101·9 (40)	100·9 (8)	94·3 (7)	99·3 (76)	99·9 (38)	—	95·2 (9)	102·1 (76)
J	130·4 (17)	137·5 (15)	134·0 (40)	131·7 (8)	126·7 (7)	133·2 (75)	131·8 (65)	124·4 (21)	126·8 (32)	137·3 (74)
NH, R	51·7 (16)	54·9 (15)	53·4 (41)	52·3 (8)	55·1 (7)	51·9 (75)	53·1 (54)	46·5 (38)	49·0 (45)	50·5 (79)
NH, L	51·6 (16)	55·1 (15)	53·5 (41)	52·4 (8)	55·2 (7)					
NB	25·3 (17)	27·1 (15)	28·1 (41)	26·6 (8)	25·5 (7)	26·1 (73)	25·5 (60)	24·0 (38)	24·2 (43)	25·6 (78)
DS	8·1 (16)	8·8 (15)	10 3 (42)	10·6 (8)	11·6 (7)	—	—	—	—	—
DC	22·1 (17)	22·7 (15)	22·8 (45)	21·3 (8)	18·7 (7)	—	—	—	—	—
DA	28·0 (16)	29·4 (15)	32·6 (41)	30·9 (8)	30·7 (7)	—	—	—	—	—
SS	2·4 (16)	2·6 (15)	3·0 (41)	3·5 (8)	3·6 (7)	—	—	—	—	—
SC	7·4 (16)	7·7 (15)	9·0 (42)	7·8 (8)	7·2 (7)	—	—	—	—	—
O_1R	41·7 (17)	44·0 (15)	44·2 (41)	44·3 (8)	44·4 (7)	—	—	—	42·6 (10)	40·9 (79)
O_1L	41·7 (17)	43·4 (15)	43·6 (41)	43·2 (8)	44·1 (7)					
O_2R	35·0 (17)	36·5 (15)	35·0 (41)	33·6 (8)	35·6 (7)	33·5 (74)	33·8 (54)	31·7 (38)	32·4 (45)	34·9 (79)
O_2L	35·0 (17)	36·7 (15)	35·0 (41)	34·3 (8)	35·9 (7)					
O_1'	39·5 (17)	41·2 (15)	39·6 (41)	39·1 (8)	39·4 (7)	39·1 (73)	38·3 (54)	36·6 (17)	37·6 (46)	—
G_1	47·7 (17)	51·5 (13)	49·9 (37)	49·2 (8)	50·0 (7)	—	54·5 (4)	50·5 (21)	46·4 (5)	53·0 (75)
G_1'	44·5 (17)	47·7 (13)	45·4 (37)	44·8 (8)	46·2 (7)	47·3 (69)	45·8 (32)	—	45·2 (7)	—
G_2	41·0 (15)	43·6 (14)	39·6 (40)	40·1 (6)	40·6 (7)	41·2 (70)	39·2 (35)	—	35·9 (9)	38·2 (74)
GL	91·5 (17)	97·2 (15)	96·6 (39)	93·8 (8)	95·9 (7)	98·2 (76)	97·7 (58)	96·0 (36)	95·1 (39)	104·9 (69)
EB	41·1 (15)	43·7 (14)	—	—	—	—	—	—	—	—
EH	11·8 (15)	14·2 (14)	—	—	—	—	—	—	—	—
fml	35·7 (17)	37·4 (15)	36·7 (43)	35·4 (8)	37·3 (8)	35·5 (73)	36·0 (23)	33·9 (21)	35·5 (8)	—
fmb	30·1 (17)	31·4 (15)	3 ·7 (42)	30·6 (8)	30·8 (8)	30·0 (74)	30·9 (23)	28·2 (21)	27·4 (8)	—
100 B/L'	79·6 (17)	75·2 (14)	82·4 (44)	80·0 (7)	79·4 (8)	—	77·8 (13)	—	76·2 (9)	—
100 H/L'	75·2 (17)	72·3 (14)	78·2 (43)	76·3 (7)	78·7 (8)	—	—	—	75·5 (9)	—
100 B/L	79·8 (17)	75·3 (14)	82·9 (44)	80·4 (7)	79·5 (8)	81·7 (77)	78·9 (73)	74·6 (21)	75·8 (33)	76·5 (88)
100 H/L	75·5 (17)	72·4 (14)	78·5 (43)	76·7 (7)	78·9 (8)	78·2 (76)	77·4 (21)	75·2 (21)	75·6 (9)	—
100 B/H	105·7 (17)	104·0 (14)	105·8 (43)	104·9 (7)	101·0 (8)	104·6 (76)	103·2 (37)	{99·1 (21)}	100·0 (43)	—
100 H'/L	75·1 (17)	72·1 (14)	{78·4 (43)}	{77·5 (7)}	{78·7 (8)}	—	77·4 (35)	—	75·8 (10)	75·6 (88)
100 B/H'	106·3 (17)	104·5 (14)	{105·7 (43)}	{104·8 (7)}	{100·9 (8)}	—	102·4 (46)	—	99·8 (10)	101·2 (88)
100 $(B-H')/L$	4·7 (17)	3·5 (14)	{4·5 (43)}	{2·9 (7)}	{0·8 (8)}	—	{1·5 (69)}	—	{0·0 (10)}	{0·9 (88)
100 $G'H/GB$	70·4 (15)	76·0 (15)	69·8 (38)	67·8 (8)	79·0 (7)	70·7 (73)	71·1 (36)	—	68·0 (8)	68·4 (65)
100 $NB/NH, R$	49·3 (16)	49·5 (15)	52·8 (41)	51·2 (8)	46·4 (7)	52·7 (41)	51·0 (8)	46·3 (7)	50·4 (73)	50·7 (78)
100 $NB/NH, L$	49·3 (16)	49·3 (15)	52·7 (41)	51·0 (8)	46·3 (7)					
100 $O_2/O_1, R$	84·3 (17)	83·0 (15)	79·1 (41)	75·9 (8)	80·3 (7)	—	—	—	77·2 (9)	85·3 (80)
100 $O_2/O_1, L$	84·2 (17)	84·6 (15)	80·0 (41)	79·4 (8)	81·5 (7)					
100 O_2/O_1'	88·7 (17)	88·6 (15)	{88·4 (41)}	{87·7 (8)}	{91·1 (7)}	86·0 (73)	88·4 (54)	84·4 (17)	86·3 (45)	—
100 fmb/fml	84·3 (17)	84·1 (15)	83·8 (42)	86·6 (8)	82·9 (8)	84·6 (73)	85·9 (23)	83·3 (21)	77·3 (8)	—
100 DS/DC	37·0 (16)	39·1 (15)	45·4 (42)	49·8 (8)	61·9 (7)	—	—	—	—	—
100 SS/SC	31·5 (16)	34·6 (15)	32·7 (41)	45·2 (8)	54·0 (7)	—	—	—	—	—
100 G_2/G_1	86·5 (15)	84·1 (12)	79·8 (37)	82·2 (6)	81·3 (7)	—	78·4 (4)	—	77·8 (5)	72·1 (72)
100 EH/EB	28·7 (15)	31·7 (14)	—	—	—	—	—	—	—	—
$P\angle$	87°·4 (15)	85°·7 (14)	86°·0 (39)	85°·9 (8)	84°·0 (7)	—	84°·9 (17)	—	86°·1 (9)	82°·0 (67)
$N\angle$	64°·7 (16)	65°·5 (15)	66°·8 (38)	65°·2 (8)	64°·4 (7)	{68°·2 (73)}	{67°·3 (49)}	—	{67°·2 (9)}	{70°·2 (69)
$A\angle$	71°·8 (16)	68°·8 (15)	70°·5 (38)	73°·4 (8)	71°·2 (7)	{70°·4 (73)}	{70°·1 (49)}	—	{74°·8 (9)}	{71°·2 (69)
$B\angle$	43°·5 (16)	45°·7 (15)	42°·6 (39)	41° 4 (8)	44°·5 (7)	{41°·4 (73)}	{42°·6 (49)}	—	{38°·0 (9)}	{38°·6 (69)
θ_1	28°·0 (15)	28°·5 (14)	27°·2 (38)	28°·8 (8)	31°·7 (7)	—	—	—	—	—
θ_2	15°·5 (15)	17°·7 (14)	15°·4 (38)	12°·9 (8)	12°·8 (7)	—	—	—	—	—
Oc. I.	61·7 (17)	59·2 (15)	62·8 (44)	61·5 (7)	62·8 (8)	—	{62·1 (19)}	—	{60·5 (9)}	—

* The capacities in square brackets were found by methods which would only give results approximately the same as those given by the method described above (p. 198). The indices enclosed in curled brackets give the ratio of the two mean measurements and are therefore only approximate values. The angles in curled brackets were determined with the Trigonometer from the mean lengths of the sides of the fundamental triangle.

a low coefficient between two races A and B indicates a closer relationship than a higher coefficient between, say, A and C, what is meant always is that it is more probable that A and B are random samples from the same population than that A and C are.

Intra-racial correlations between characters not connected in some obvious way, such as the various measurements of head length, are known to be small; otherwise the coefficient could not theoretically be used. The measurements chosen should therefore show little correlation among themselves and at the same time be as representative as possible of all regions of the head. Further, the same list should as far as possible be used for computing all coefficients.

The standard deviations σ_s and σ_s' cannot be determined with at all sufficient accuracy from the short series of skulls with which, as a rule, the craniometrician is concerned. Accordingly the assumption has to be made that they are equal to each other and to the standard deviation of the longest homogeneous series of crania available. This supposes that the races considered are alike both in variability and homogeneity. This latter condition may seldom be closely fulfilled, but we are obliged to postulate it as long as the inter-racial variabilities are unknown. At least we have some indication from the standard deviation of the cephalic index that all the modern Asiatic races we shall deal with are of a similar degree of heterogeneity and this makes the coefficients derived from them more comparable.

If $\sigma_s = \sigma_s'$ then:

$$\text{C. L.} = \frac{1}{m} S \left\{ \frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s} \right)^2 \right\} - 1 \pm \frac{\cdot 67449}{\sqrt{2m}},$$

and we shall use the coefficient in this form.

Several series of sufficient length for which the variabilities of characters are given were found. These were:

(i) An unpublished series (E) of 800 male Egyptian crania; (ii) Naqada Prehistoric Egyptians measured by Fawcett (*Biometrika*, Vol. I. p. 438); (iii) Whitechapel English—Macdonell (*Biometrika*, Vol. III. p. 222); (iv) Moorfields English—Macdonell (*Biometrika*, Vol. V. p. 92); (v) Congo Negroes—Benington (*Biometrika*, Vol. VIII. p. 298).

A table of standard deviations compiled from these sources is given below.

The variabilities of the 5 races shown in Table II are in fairly close agreement and the Egyptians and Negroes do not seem to differ appreciably from the English. This fact to a certain extent justifies the invariable use of the standard deviations of one race only in computing Coefficients of Racial Likeness, but this use is still open to criticism if any of the data compared are suspected to be less homogeneous than the average. The standard deviations of the Egyptian Series E were used in calculating all Coefficients of Racial Likeness since the probable errors are smaller than those of the other races.

The need has been noted above of using, for the computation of the Coefficient of Racial Likeness, only characters which will be fairly independent of each other.

TABLE II.

Standard Deviation of Characters. All Male.

Race	*B/L'*	*H/L'*	*B/H*	*Oc. I*	*G'H/GB*	O_2/O_1 *(R)*	*NB/NH (R)*	*fmb/fml*	G_2/G_1	$\hat{P}$	$\hat{N}$
Egyptians (E) ...	2·67 ± ·04	2·94 ± ·05	4·30 ± ·06	3·30 ± ·05	4·96 ± ·08	5·05 ± ·08	3·82 ± ·06	5·79 ± ·09	6·79 ± ·13	3·24 ± ·05	3·31 ± ·06
Naqada	2·88 ± ·13	2·77 ± ·13	4·53 ± ·19	—	4·52 ± ·25	5·00 ± ·27	4·18 ± ·23	—	7·36 ± ·43	2·87 ± ·17	3·77 ± ·21
Whitechapel English	2·97 ± ·17	2·67 ± ·15	5·14 ± ·23	—	5·39 ± ·35	4·66 ± ·27	4·58 ± ·26	—	6·40 ± ·39	3·92 ± ·24	3·52 ± ·20
Moorfields English	3·27 ± ·37	3·13 ± ·36	—	—	4·77 ± ·61	4·16 ± ·48	4·13 ± ·46	6·64 ± ·54	5·59 ± ·74	3·99 ± ·49	3·54 ± ·41
Congo Negroes ...	2·88 ± ·13	—	4·55 ± ·31	—	4·82 ± ·34	5·12 ± ·36	4·91 ± ·33	5·96 ± ·41	6·27 ± ·55	3·74 ± ·26	—

Race	$\hat{A}$	*F*	*B*	*B'*	*OH*	*LB*	*Q*	*S*	*U*	*G'H*	*J*
Egyptians (E) ...	3·46 ± ·06	5·73 ± ·09	4·76 ± ·08	4·05 ± ·06	4·12 ± ·07	3·97 ± ·06	9·75 ± ·16	12·51 ± ·20	13·77 ± ·22	4·15 ± ·07	4·57 ± ·08
Naqada	4·16 ± ·23	6·03 ± ·26	4·60 ± ·19	4·82 ± ·19	4·46 ± ·18	4·85 ± ·22	10·11 ± ·45	11·91 ± ·52	13·00 ± ·57	4·11 ± ·21	5·22 ± ·34
Whitechapel English	3·41 ± ·20	6·17 ± ·25	5·28 ± ·22	4·20 ± ·17	4·28 ± ·18	4·13 ± ·18	11·40 ± ·51	13·69 ± ·57	15·02 ± ·63	3·86 ± ·21	5·57 ± ·40
Moorfields English	3·71 ± ·43	5·90 ± ·42	5·31 ± ·37	4·12 ± ·29	4·69 ± ·33	4·57 ± ·37	12·54 ± 1·06	12·01 ± ·90	14·45 ± 1·13	4·08 ± ·43	4·65 ± ·84
Congo Negroes ...	—	6·55 ± ·44	5·00 ± ·34	3·85 ± ·26	4·05 ± ·28	4·58 ± ·32	10·29 ± ·72	10·73 ± ·72	17·24 ± 1·16	4·14 ± ·28	6·90 ± ·57

Race	*NH, R*	*NB*	O_1R	O_2R	G_1	G_2	*fml*	*fmb*	*C*
Egyptians (E) ...	2·92 ± ·05	1·77 ± ·03	1·67 ± ·03	1·91 ± ·03	3·33 ± ·06	2·63 ± ·05	2·47 ± ·04	2·15 ± ·03	113·51 ± 1·97
Naqada	3·00 ± ·05	1·98 ± ·10	2·14 ± ·11	2·32 ± ·13	3·62 ± ·20	3·75 ± ·21	—	—	106·68 ± 5·42
Whitechapel English	2·60 ± ·14	2·16 ± ·12	2·02 ± ·12	2·22 ± ·13	2·74 ± ·16	2·85 ± ·17	—	—	122·37 ± 6·88
Moorfields English	2·60 ± ·28	1·90 ± ·21	1·42 ± ·16	2·12 ± ·24	4·25 ± ·49	2·36 ± ·29	2·86 ± ·23	1·82 ± ·15	132·18 ± 13·44
Congo Negroes ...	2·83 ± ·19	2·05 ± ·14	1·64 ± ·12	1·95 ± ·14	4·49 ± ·39	3·95 ± ·28	2·39 ± ·16	2·30 ± ·16	126·57 ± 8·81

For example, if the sagittal arc S is employed, the arcs S_1, S_2, S_3 and the chord S_3' may not be for they are significantly correlated with S; two angles of the fundamental triangle may be included but not three. The correlations between the arcs and lengths of the skull not connected in any such close way and those between an index and its component lengths are mostly small, although some are quite significant, and so provisionally we may use the 31 measurements, for which the standard deviations are given in Table II*. The correlations between these, if sensibly differing from zero, are fairly small and it does not seem possible to advantageously add to the list. If the coefficients computed by different workers are to be comparable, some such list must be adopted and adhered to. Unfortunately all these mean measurements will not be given for some comparative material, but the majority are standard ones adopted by most craniometricians. Our most meagre data are those for the Maravar crania for which there are only 19 of the 31 chosen characters. For all the Asiatic material the transverse arc was given only measured to the auricular points (Q') and not to the ear-rods (Q). But we were only provided with the standard deviation of Q and this was used instead of that of Q'; the difference between these two variabilities would certainly be small. In comparing the Tibetan means with those of unallied races Q and not Q' was used. Besides the Coefficient of Racial Likeness computed from the 31 characters, we have computed the coefficient for indices and angles alone, using the 12 given in Table II.

The results for the two Tibetan and other races are given below (Table III).

We shall first consider the relations between the two Tibetan series and other Asiatic races as shown by the Coefficients of Racial Likeness computed from all the characters—lengths, arcs, indices and angles. There is clearly a remarkable difference between the two types: the coefficients with Type B are in every case larger and in most cases approximately twice as large as those with Type A. Our supposed hybrids (A) are seen to be very similar to the Burmese hybrids and more akin to all the Burmese, the Malayans and the Chinese than to their own countrymen of Khams†. This is distinct evidence in support of the hypothesis which would make Type A a mixed race. To the aboriginal Dravidian (Maravar), who rarely leaves the confines of his own land, the western Tibetan has no relationship but the closer affinity to the Hindu is, perhaps, due to or symptomatic of contact through trade. Our Hindu data are representative of the Bengali, a supposed eastern people, and not of migrants from the west. There is little direct contact in these days between western Tibet and Burma and certainly much less than between China and Tibet, so our figures are quite consistent with the philological and historical evidence which tells of Tibeto-Burmese invasions from north-west China and they show that all these oriental peoples are very similar in type.

The Tibetan Type B is most nearly, though not closely, related to the Burmese

* These 31 characters were chosen so that no pair would be highly correlated and there was no other method of selection. They include all the more important measurements of the skull.

† For the Moorfields and Whitechapel English measured by Macdonell I find C. L. $= 2{\cdot}05 \pm {\cdot}09$ using 26 characters and $1{\cdot}73 \pm {\cdot}16$ for indices and angles only (9 characters), so the coefficient for Tibetan A and Burmese B must be supposed to indicate quite a close relationship.

TABLE III.

Coefficients of Racial Likeness. All Male.*

		Tibetan, Type A						Tibetan, Type B (Khams)					
		All Characters			Indices and Angles only			All Characters			Indices and Angles only		
		C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters	C. L.	Probable Error	Number of Characters
Asiatic Races	Burmese B (Hybrids ?)	·80	± ·09	31	·33	± ·14	12	4·83	± ·09	31	4·11	± ·14	12
	Burmese C (Karens ?)	2·36	± ·09	31	2·27	± ·14	12	3·46	± ·09	31	3·67	± ·14	12
	Malayan	2·88	± ·10	21	1·13	± ·21	5	11·17	± ·10	21	15·99	± ·21	5
	Chinese	3·23	± ·10	24	·82	± ·18	7	6·37	± ·10	24	2·21	± ·18	7
	Burmese A (Burmans)	5·71	± ·09	31	4·87	± ·14	12	10·96	± ·09	31	14·07	± ·14	12
	Tibetan B	6·46	± ·09	31	3·44	± ·14	12	—	—	—	—	—	—
	Hindu	6·56	± ·09	28	5·71	± ·16	9	18·34	± ·09	28	4·78	± ·16	9
	Maravar	12·34	± ·11	18	10·96	± ·21	5	25·72	± ·11	18	3·65	± ·21	5
	Tibetan A	—	—	—	—	—	—	6·46	± ·09	31	3·44	± ·14	12
	Aino	13·82	± ·10	23	8·84	± ·17	8	13·61	± ·10	23	11·94	± ·17	8
Other Races	Congo Negroes ...	7·23	± ·09	29	11·55	± ·15	10	21·61	± ·09	29	18·68	± ·15	10
	French	9·12	± ·10	25	9·47	± ·17	8	22·96	± ·10	25	16·96	± ·17	8
	Moriori...	12·78	± ·09	30	7·38	± ·14	11	4·85	± ·09	30	4·74	± ·14	11
	Whitechapel English...	14·59	± ·10	24	11·97	± ·16	9	11·93	± ·10	24	5·32	± ·16	9
	Naqada (Egyptians) ...	17·64	± ·09	27	23·57	± ·15	10	18·32	± ·09	27	11·57	± ·15	10

* The references to the tables of means for the races other than Asiatic are:

Congo Negroes. Biometrika, Vol. VIII. p. 298. This is the series which was measured by Dr Benington.

Whitechapel English and French (the latter quoted by Macdonell). *Biometrika*, Vol. III. p. 208.

Moriori. Biometrika, Vol. XI. p. 93.

Naqada and Aino (quoted by Fawcett). *Biometrika*, Vol. I. p. 426.

hybrids and Karens. It has equal affinities with the Chinese and Tibetan A. The coefficient with the Maravar is the highest recorded in the table, for both Asiatic and western races, and it must indicate a very fundamental difference in type. The Tibetan B is markedly differentiated, too, from the Hindu, Malayans and Burmans. If, as the close relationships of our Type A with these peoples suggested, they had all a common primitive origin then the Type B must belong to a radically different stock. Though adjacent to the Chinese it is much less like that people than are the inhabitants of Sikkim 600 miles away. The fact that the Tibetans of Khams resemble none of their neighbours suggests that they belong to a well differentiated and possibly pure stock.

We turn now to the coefficients between Tibetan A and the races other than Asiatic. These all appear to be smaller than we should have anticipated when compared with the earlier ones considered. It is surprising to find that two Tibetan peoples are hardly more alike in cranial type than one of them is to a race of African negroes and that Frenchmen have heads that more closely resemble those of a Tibetan people than the latter do those of Indians living not a 1000 miles from them.

Turning to the coefficients with Tibetan B, the outstanding feature is the lowness of that with the Moriori. Even among Asiatic races there are only two, Burmese B and C, smaller than it. Only more abundant comparative material can show what the significance of this somewhat unexpected similarity between two primitive peoples is*.

The primitive Aino are almost as far removed from the two Tibetan types as any of the races.

If mere size is to have no part in determining racial affinities then we must base judgments on indices and angles only. To test the accordance of such a method with the more general one, which considers both size and form, the Coefficients of Racial Likeness were computed without including lengths and arcs and these are given in Table III.

In most cases these coefficients were lower than those previously found—there being 21 such cases out of a total of 28—and many of the differences were very considerable. Looking first at the coefficients for Tibetan A and other Asiatic races: every one of these deduced from indices and angles only is smaller than the corresponding one for all the characters and the races are now arranged in a much more suggestive order. After the Burmese hybrids, the Chinese have cranial indices and angles most similar to those of the Tibetan A type and the resemblance is very close indeed. It is noteworthy, that the corresponding coefficient with Tibetan B is the lowest of all with that type, so the Chinese skull is very similar in shape if not in size to that of both our series. The coefficient for Tibetan A and Tibetan B is lessened quite considerably by taking fewer characters, but there is a far greater reduction than that in the case of the Hindu and Maravar with Tibetan B. There would seem to be a great difference between the Tibetan B

* Our coefficients are unfortunately not comparable with those given by Miss Tildesley in her Burmese paper as I have not used precisely the same characters as she did.

type and those of the two Indian peoples in size, the Tibetan being much larger than the others, but comparatively little in shape of head as characterised by angles and indices. But in one case the coefficient was calculated from only 9 characters and in the other there were only 5 available. We are dealing here with small samples and no stress can be laid on conclusions deduced from them. It may be that there are some characters which usually differ more significantly between race and race than other characters do and if these are neglected when a coefficient is being calculated from a small number of means we may arrive at very misleading results.

If the Coefficients of Racial Likeness between the two Tibetan and non-Asiatic races are compared, the values given for indices and angles only are not, with one exception, viz. that of the Whitechapel English and Tibetan B, smaller than the coefficients for all characters and in this respect there appears to be a very real difference between the allied and unallied races. Further evidence is needed to establish definitely the theory, but it seems at present to be highly probable that differences in size are of relatively little importance; resemblance between the shapes of heads is the real criterion of relationship and this we are able to measure with angles and indices.

7. *Comparison of the Tibetan mean direct Measurements with those of other Races.*

The means of samples taken even from a very skew frequency distribution are known to have themselves a distribution which is nearly normal, so that if the difference of two mean characters divided by the probable error of that difference (i.e. approximately $\dfrac{M_s - M_s'}{\cdot 67449\sqrt{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}}}$, with the notation of p. 205) is between 0 and 2 then the means might very probably have been random samples from the same population; if between 2 and 3, the inference is probable but uncertain; while a value over 3 denotes great improbability (of the order ·002). Hence if the value of $\left\{\dfrac{(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}}\right\}$ is between 0 and 2·7 a verdict can be given in favour of common origin; between 2·7 and 6·1 it is uncertain and greater than 6·1 very improbable. After making the assumption that the two standard deviations may be considered equal, this function, i.e. $\dfrac{n_s n_s'}{n_s + n_s'}\left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2$ which we will call α, becomes one which has already been evaluated for the individual characters of the Tibetan and other races, when computing the Coefficients of Racial Likeness. We have C.L. $= \dfrac{1}{m}S(\alpha) - 1$. From the values of α we can see at once in which characters the two races compared differ most essentially and in which they are most alike. Tables IV A and IV B give these values of α.

All the Burmese types, the Malayan and the Chinese are more closely related to the Tibetan A race than the Tibetan B is. In what ways do these two differ

TABLE IV A.

Values of $\frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s}\right)^2$ *for Tibetan A and other Races.*

		Asiatic Races									Other Races				
		Burmese B (Hybrids?)	Burmese C (Karens?)	Malayan	Chinese	Burmese A (Burmans)	Tibetan B	Hindu	Maravar	Aino	Congo Negroes	French	Moriori	Whitechapel English	Prehistoric Egyptians (Naqada)
Indices	B/L'	·11	·03	8·55	3·34	13·49	20·82	9·54	32·94	19·21	4·55	13·24	15·28	37·03	97·16
	H/L'	·69	7·71	—	—	12·69	7·48	·06	·00	·26	·06	12·69	5·25	36·38	6·71
	B/H	·17	6·79	·90	3·94	·01	1·23	21·39	22·13	36·67	2·99	11·28	·88	·28	28·32
	Oc. I.	·02	·60	—	—	1·36	4·57	—	—	—	58·06	—	—	—	—
	$G'H/GB$	1·43	14·35	·05	·21	·16	9·56	1·22	—	1·99	5·63	6·94	5·80	12·36	·03
	$O_2/O_1(R)$	9·01	3·11	—	—	14·44	·53	11·63	—	·55	·02	4·60	·04	23·23	48·14
	$NB/NH(R)$	·29	3·00	1·09	·25	9·66	·02	·13	4·45	1·78	28·76	13·64	21·73	2·58	2·94
	fmb/fml	·86	·32	·04	·49	·10	·01	7·96	·28	—	·24	—	3·16	—	—
	G_2/G_1	1·72	2·82	—	4·49	10·40	·83	7·65	—	8·44	20·41	10·51	39·94	—	56·67
Angles	$P\angle$	·50	·37	—	·00	1·25	1·99	·77	—	9·82	4·78	2·89	·04	1·82	·29
	$N\angle$	·07	·04	—	—	4·53	·46	—	—	—	—	—	·01	·30	4·32
	$A\angle$	1·14	·15	—	—	1·59	5·82	—	—	—	—	—	·00	2·78	1·10
Lengths	F	·66	·20	—	6·79	1·50	23·87	·85	·37	—	1·12	10·78	20·15	77·98	41·89
	B	·63	·24	4·80	·00	10·06	·00	30·29	33·11	2·04	·44	9·19	1·99	1·12	13·51
	B'	2·60	1·61	·54	1·30	2·16	1·40	·01	·21	11·90	18·57	10·30	5·09	26·78	2·08
	OH	4·67	5·64	23·88	29·19	18·04	2·48	·62	—	36·50	1·06	·03	12·91	3·20	7·50
	LB	3·31	8·48	12·70	9·90	6·04	6·18	9·50	7·14	84·85	4·97	13·21	70·30	32·77	12·06
Arcs	Q' or Q^*	3·75	4·02	9·34	12·85	27·66	·31	4·12	—	—	·32	·22	2·90	—	6·81
	S	1·18	1·47	2·03	7·89	·61	15·37	·48	—	12·60	·06	2·16	4·52	25·23	13·92
	U	·91	·00	·32	1·79	·16	19·60	7·34	13·75	26·84	·05	14·66	22·03	34·01	4·29
Face	$G'H$	·44	7·22	·36	2·16	2·52	21·98	10·25	—	·11	24·04	·69	30·35	·46	2·41
	J	·44	3·25	5·20	1·27	7·40	19·21	6·83	16·19	31·21	8·17	·05	26·56	·05	14·18
Nose	$NH(R)$	·22	6·99	·07	2·84	3·90	9·29	10·09	35·69	2·25	28·64	·37	40·36	·40	12·50
	NB	2·94	·06	2·83	·17	30·10	8·25	4·71	6·34	·40	1·99	20·16	·00	4·36	·17
Orbit	O_1R	13·18	12·96	—	—	26·92	15·11	1·83	—	3·20	1·60	57·26	29·90	8·24	4·07
	O_2R	2·87	·49	8·52	5·11	·00	4·91	22·85	35·05	·04	·55	10·33	16·42	9·56	36·21
Palate	G_1	1·10	2·37	—	—	5·08	9·59	·59	6·64	35·10	6·62	1·43	29·50	—	81·99
	G_2	·50	·11	·07	4·91	3·09	7·07	21·14	—	14·13	7·21	24·69	6·63	31·15	·89
For. Mag.	*fml*	·08	8·76	·10	·15	1·48	3·78	·04	4·99	—	·09	—	·29	—	—
	fmb	·29	·88	·03	1·36	·94	2·92	8·58	7·34	—	·03	—	1·20	—	—
Capacity	C	·08	·02	·08	1·15	·56	6·53	11·23	13·55	·90	7·69	1·62	·11	2·06	2·99
C. L.		·80	2·36	2·88	3·23	5·71	6·46	6·56	12·34	13·82	7·23	9·12	12·78	14·59	17·64

* See p. 209.

TABLE IV B.

Values of $\frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s}\right)^2$ *for Tibetan B and other Races.*

		Asiatic Races									Other Races				
		Burmese C (Karens?)	Burmese B (Hybrids?)	Chinese	Tibetan A	Burmese A (Burmans)	Malayan	Aino	Hindu	Maravar	Moriori	Whitechapel English	Prehistoric Egyptians (Naqada)	Congo Negroes	French
Indices	B/L'	12·60	15·08	6·39	20·82	77·22	69·70	2·86	·77	·43	2·35	·00	10·78	12·03	80·88
	H/L'	24·14	8·64	—	7·48	42·54	—	15·22	6·49	8·17	·93	5·12	1·15	12·05	·06
	B/H	2·49	·21	·36	1·23	1·85	·23	17·66	9·12	10·90	·13	3·57	12·07	·10	19·67
	Oc. I.	6·20	2·31	—	4·57	13·30	—	—	—	—	—	—	—	96·63	—
	$G'H/GB$	1·75	14·26	10·33	9·56	15·64	14·21	30·20	13·58	—	1·53	·12	14·84	38·07	5·53
	$O_2/O_1\,(R)$	1·37	5·75	—	·53	6·54	—	2·61	7·43	—	·40	13·54	32·23	·53	8·19
	$NB/NH\,(R)$	3·35	1·03	·29	·02	8·19	·70	1·25	·03	3·57	22·36	2·92	2·20	25·55	14·29
	fmb/fml	·22	·98	·88	·01	·03	·10	—	7·20	·17	3·32	—	—	·13	—
	G_2/G_1	·75	·31	2·12	·83	3·63	—	32·13	3·04	—	22·86	—	32·86	11·56	3·43
Angles	$P\angle$	·28	3·53	2·11	1·99	7·69	—	1·59	4·39	—	·99	9·17	1·56	·16	11·61
	$N\angle$	·53	·05	—	·46	1·63	—	—	—	—	·47	·10	1·37	—	—
	$A\angle$	2·30	9·22	—	5·82	2·55	—	—	—	—	7·83	21·78	16·64	—	—
Lengths	F	12·57	22·73	5·47	23·87	47·41	—	—	24·08	18·54	1·91	3·25	·10	22·42	10·36
	B	·22	·59	·01	·00	8·70	4·10	1·73	25·81	28·84	1·75	9·43	11·35	·39	7·90
	E'	4·84	6·44	·11	1·40	·00	·94	2·82	1·32	·65	·64	11·25	8·45	7·20	2·60
	OH	1·11	·81	11·64	2·48	4·80	7·09	13·94	4·57	—	1·58	·12	·28	·81	3·06
	LB	·51	·05	·01	6·18	·34	·08	31·18	·00	·11	26·98	4·86	·01	·72	·19
Arcs	Q or Q^*	2·20	2·08	7·25	·31	17·97	4·28	—	5·96	—	·91	—	9·67	1·43	·00
	S	4·08	4·01	·63	15·37	15·07	6·60	2·72	14·37	—	6·09	·18	2·51	19·72	11·36
	U	13·59	19·16	18·44	19·60	22·19	24·75	·62	63·15	73·46	·41	·11	14·07	26·17	3·24
Face	$G'H$	1·11	20·89	18·75	21·98	16·37	29·59	32·48	52·81	—	·01	28·53	58·71	114·58	45·13
	J	26·66	8·40	18·95	19·21	6·39	11·06	·02	60·88	71·85	·00	29·13	79·22	59·70	26·17
Nose	$NH\,(R)$	·05	4·13	4·46	9·29	2·90	13·19	28·62	45·91	88·92	7·09	20·22	54·34	79·83	18·99
	NB	3·90	·42	9·82	8·25	3·51	3·97	9·05	21·69	33·00	10·88	29·46	16·32	4·47	60·47
Orbit	O_1R	·27	·17	—	15·11	·16	—	43·42	4·11	—	·60	4·42	8·90	34·09	139·95
	O_2R	1·06	12·02	23·44	4·91	6·76	30·76	8·84	51·81	67·95	1·82	32·44	71·96	11·19	33·18
Palate	G_1	·92	2·36	—	9·59	·22	—	2·25	8·47	·72	2·15	—	18·47	1·18	22·84
	G_2	6·07	7·43	27·98	7·07	23·98	9·71	49·61	46·95	—	31·66	77·18	18·57	34·39	66·29
For. Mag.	fml	·01	3·42	2·91	3·78	·89	7·37	—	3·09	17·57	7·46	—	—	4·22	—
	fmb	·41	·73	·49	2·92	·12	5·29	—	18·08	19·39	·81	—	—	4·85	—
Capacity	C	2·61	3·63	4·09	6·53	12·24	11·77	5·26	36·51	36·64	9·68	3·36	23·02	31·44	3·67
C. L.		3·46	4·83	6·37	6·46	10·96	11·17	13·61	18·34	25·72	4·85	11·93	18·32	21·61	22·96

* See p. 209.

most? Of the 31 values of α given there are 15 greater than 6·1. The largest of these are for the characters B/L', F, S, U and $G'H$, the B type being in every case the larger. Its greater size, weight and capacity are thus chiefly due to its excess length; in breadth and height, as shown by the low values of α for B, B', OH and Q, the two are not significantly different. Another very noticeable difference between them is for the size and shape of the face ($G'H$, J and $G'H/GB$), the B type being again the larger. The orbits, nose and palate differ in the same way in all dimensions, though they are remarkably similar when measured indicially. In fact these two races apparently resemble each other in both size and shape only in the neighbourhood of the transverse section through the auricular points and in the *foramen magnum*. The Eastern Tibetan is altogether a bigger type than its compatriot one of the West.

We turn now to compare the races more like the latter with it, and we take first the Burmese B. This series only consisted of 8 skulls and no great reliance can be placed on deductions drawn from its mean values, but it seems to be remarkably closely related to the Tibetan A. Both these, it is interesting to notice, have been supposed to represent hybrid stocks. Only for one direct measurement do they differ by more than the amounts we should expect from random sampling. That is in the width of orbit for which the Burmese exceeds the other by 2·6 mm. The three Burmese types are very alike in that particular. The C is distinguished from the Tibetan by a greater upper face height ($G'H$), giving a very different facial index ($G'H/GB$), and OH height (affecting the indices H/L' and B/H). The excess in alveolar-basion and foraminal lengths is also noticeable.

The Burmese A series is the only one with which reliable comparison can be made since the others are based on such short series. The mean breadth of the calvaria and face and the OH height are appreciably greater than for the Tibetans, though the index B/H is almost the same for both and the measurements made in the sagittal plane are very similar. Thus the discrepancies between the lengths and arcs of the brain-box are almost the exact reverse of those between the Tibetan B and A types. The Burmese A differs most significantly from the latter with its much smaller nasal breadth which gives the very high value $\alpha = 30{\cdot}1$. The difference in palate index is well marked.

The characters for which the α's with the Malayan type are greater than 6·1 are B/L', LB, Q', O_2R and most notably OH, the Tibetan mean being 5·4 mm. less than the other; and with the Chinese we find F, LB, Q', S and OH, the last showing again the highest value. We should expect to find the Malayans and Chinese quite closely related since they appear to resemble and differ from the Tibetans in very much the same way. The comparison of the Chinese with our A type proves particularly interesting. For all the indicial measurements and angular measurements and for those of the face, nose, orbit (though O_1R is not available), palate and *foramen magnum* the value of α is less than 6·1 and for 9 of the 15 of these it is less than 2·7. All the significant differences are in the lengths of the calvaria. In breadth the two only differ by 0·1 mm. but in length (F) the Chinese exceeds the Tibetan by 5·2 mm., for LB 3·4 mm., for Q' 10·3 mm., and for S 9·4 mm.

The lesser breadth of the Hindu skull is the feature which distinguishes it most from that of its northern neighbour. In size, as shown by the longer lengths and arcs, the two do not differ greatly, but all the facial measurements are characteristic of each. The Maravar is evidently a much more divergent type than any we have considered yet. Its points of greatest difference and similarity with the Tibetan appear to be scattered fortuitously throughout the short list. The characteristic nose of the Dravidian finds no counterpart here.

The smallness of many of the mean measurements of the Tibetan A type has been repeatedly noticed in comparing it with these other Oriental peoples and that has accounted for many of its greatest divergencies from them. Consulting the table of means (Table I), we find that it is smaller than all the others in the measurements *LB*, *S*, and O_1R (but not O_2R), and if the Hindu and Maravar are excluded the following can be added: *OH*, *Q'*, *NH*, *NB*, G_1 and *B* (this last being equal to the Tibetan B breadth). The smallness of the transverse vertical section should be most apparent. The Indian peoples appear to resemble the Tibetan A most in absolute size. The Coefficients of Racial Likeness between the Tibetan A and other Oriental races computed for indices and angles alone were found to be all smaller than those derived from the complete list of characters; the reductions were greatest in the case of the Malayans and Chinese. Now it has been shown by comparing the individual characters that the Tibetan only differs significantly from these two in having a smaller calvaria and to show how similar the other measurements are I have calculated the coefficients from them alone*. These are given in the following table:

TABLE V.

Comparison of Coefficients of Racial Likeness for various sets of characters. Tibetan A with other Races.

	All Characters		Angles and Indices only		All characters except those of calvaria	
	C. L.	Number of Characters	C. L.	Number of Characters	C. L.	Number of Characters
Burmese B (Hybrids?)	·80 ± ·09	31	·33 ± ·14	12	1·14 ± ·11	18
Burmese C (Karens?)	2·36 ± ·09	31	2·27 ± ·14	12	2·74 ± ·11	18
Malayan	2·88 ± ·10	21	1·13 ± ·21	5	·67 ± ·14	11
Chinese	3·23 ± ·10	24	·82 ± ·18	7	·80 ± ·13	13
Burmese A (Burmans)	5·71 ± ·09	31	4·87 ± ·14	12	5·86 ± ·11	18
Tibetan B	6·46 ± ·09	31	3·44 ± ·14	12	5·74 ± ·11	18
Hindu...	6·56 ± ·09	28	5·71 ± ·16	9	6·26 ± ·12	16
Maravar	12·34 ± ·11	18	10·96 ± ·21	5	12·00 ± ·16	9

The Chinese, Malayans and Western Tibetans all have similar skulls, but the calvariae differ more than the faces. In spite of their very close resemblance the Tibetan A is smaller for nearly all the direct measurements than the other

* That is to say, the measurements used were $G'H/GB$, O_2/O_1 (*R*), NB/NH, *fmb*/*fml*, G_2/G_1, $\hat{P}$, $\hat{N}$, $\hat{A}$, $G'H$, *J*, NH (*R*), *NB*, O_1R, O_2R, G_1, G_2, *fml*, *fmb*—18 in all.

two; it exceeds the Chinese only in the lengths G_2 and O_2R and the Malayans in L, O_2R, *fml*, *fmb* and capacity. To produce the Chinese or Malayan skull from the Tibetan we should have to make the latter increase in all directions at the same time, the rate of increase being, perhaps, greater in proportion for the calvaria than for the face.

There are 25 indicial and direct measurements given for the Maravar (Dravidian) race in the Comparative Table of Means (Table I) and if these are compared with the Tibetan A and Hindu (Bengali) means the latter will be found, for the majority of the characters, to be intermediate in size between the other two. There are 8 exceptions to this rule, viz. for H/L, *fmb*/*fml*, F, B', LB, S_1, G_1 and *fmb*. The so-called Hindu is apparently more akin to the Dravidian than to the Tibetan type.

Turning to Table III we find that every single Coefficient of Racial Likeness of the Tibetan B type with any Asiatic race other than the Aino is higher than the corresponding one of the Tibetan A, but again the Burmese series B and C have the lowest values. For the latter of these α is over 6·1 for the characters B/L', H/L', *Oc. I.*, F, U and J and for the former, B/L', H/L', F, U, J, $G'H/GB$, $A\angle$, B', $G'H$, O_2R and G_2. They both evidently differ from the Tibetan in much the same way. Their lengths (F) are very significantly less, but the breadths are much the same for all three types.

All the greatest differences between the Tibetan B and Chinese are in the facial region, the most marked being between the face heights ($G'H$) and zygomatic breadths (J), the heights of the orbits (O_2R) and the lengths of the palate (G_2). The Chinese is the smaller in all these cases as it is in horizontal circumference (U), but this order is reversed for the characters OH and Q. The Tibetan B type has a longer head than any of the Asiatic races we are able to compare it with, but, at the same time, it is unusually low in proportion to its length so that its indices H/L', H/L, and H'/L are considerably lower than any of the others. Its cephalic index, too, is, with the exception of that of the Maravar, the lowest shown.

The Burmese A type appears to diverge much more widely from the Tibetan B than any we have yet compared with the latter. Of the 31 values of α given there are 17 greater than 6·1 and 5 of these are greater than 20. The difference in length (F), which is reflected in the indices B/L' and H/L', is extreme and actually far greater than any between the Tibetan and quite unallied races. The greater differences between the Hindu and Maravar and the Tibetan B are confined almost entirely to the direct measurements. When angles and indices only are considered the Coefficients of Racial Likeness are lowered in a very remarkable way (see Table III) and we cannot yet say definitely what the significance of such a change is. The difference in size appears to be relatively greater for the face than for the calvaria.

In absolute size the Tibetan A crania were shown to be smaller than those of any other of the Asiatic races, with the possible exception of the Hindu and Maravar. The B type is at the other extreme, being, without doubt, larger than

any of the others. It exceeds them for the characters *F*, *L'*, *B'* (with Burmese A), S_1, S_1', S_3, S_3', *U*, *G'H*, *J*, O_2R, G_2, *fml*, *fmb* and *C*.

A comparison of the Tibetan means with those of the races other than Asiatic shows that the high and low values of α are scattered more or less at random throughout the list of characters. Two columns require particular mention. The first is that showing the relationships between the Tibetan A people and the Congo Negroes; α has a value greater than 6·1 for the measurements *Oc. I.*, *NB/NH (R)*, G_2/G_1, *B'*, *G'H*, *J*, *NH (R)*, G_1, G_2 and *C*. The chief differences are thus between the noses, palates and face breadths, but far more significant than any of these is the occipital index. That character is given for so few of the races that we can, at present, infer very little from it. But except for the lower occipital region, the calvariae of the Negroes and Tibetans are very similar both in size and shape.

The Coefficient of Racial Likeness for the Tibetan B type and Moriori is remarkably low. The characters giving values of α over 6·1 are *NB/NH (R)*, G_2/G_1, *LB*, *NH*, *NB*, G_2, *fml*, and *C* which show that all the significant differences are confined to the nasal regions and the base of the skull from opisthion to alveolar point.

Having considered the vertical arrays of Tables IV A and IV B we may ask if the horizontal ones are of any interest. These may show, if we consider both the Asiatic and the other races, what measurements are most likely to characterise the racial skull and in what ways all races resemble each other most closely. Or, we may hope to discover, if there are any characters which are alike for all the members of a family of races, while differing from unallied races. But far more ample and reliable comparative material than we have at present will be needed before any definite answers can be given to such questions.

We will consider first the relationships of the Tibetan A mean measurements with those of other races. There seems to be no indicial or angular measurement here which has not high and low values of α for both of the two racial groups (Asiatic and other races). The palatine index (G_2/G_1) is the only one which affords any clear distinction between them, being, apparently, fairly consistently lower (i.e. in the value of α) for the Asiatics. The corresponding array of Table IV B confirms this idea. The following table of the index shows that the squarer palate of

Values of the Palatine Index for Various Races.

Race	100 G_2/G_1	G_2	G_1
Prehistoric Egyptians (Naqada)	71·9	40·1	55·8
Aino	72·1	38·2	53·0
Modern French	72·2	37·2	46·6
Moriori	73·2	38·9	53·1
Modern German	75·4	33·2	44·3
Whitechapel English	76·3	36·8	48·3
Congo Negroes	76·8	38·9	50·3
Hindu (Bengali)	77·8	35·9	46·4
Chinese	78·4	39·2	54·5
Burmese A	79·8	39·6	49·9
Burmese C	81·3	40·6	50·0
Burmese B	82·2	40·1	49·2
Tibetan B	84·1	43·6	51·5
Tibetan A	86·5	41·0	47·7

all the Asiatics but the Aino, does distinguish them from most other races. The index itself provides a much more suggestive order than either of its components (G_2 and G_1) alone do.

The only absolute measurement which appears to be at all likely to differentiate Asiatics from the other races is the forehead breadth (B'). The latter, with the exception of the Naqada Egyptians, are all larger than the others in that particular. B' does not seem to be at all highly correlated with the zygomatic breadth, J. The Aino, again, show no resemblance to other Asiatic peoples.

Values of B' and J for Various Races.

Race	B'	J
Modern German	103·7	135·0
Whitechapel English	98·0	130·4
Congo Negroes	97·5	126·5
Aino	96·2	137·3
Modern French	96·2	130·7
Moriori	95·3	137·4
Tibetan B	94·3	137·5
Burmese A	94·3	134·0
Chinese	93·9	131·8
Malayan	93·4	133·2
Maravar	93·2	124·4
Tibetan A	92·6	130·4
Hindu	92·4	126·8
Prehistoric Egyptians (Naqada)	91·1	125·6
Burmese C	90·4	126·7
Burmese B	89·7	131·7

There seems to be no other character which at all rigidly distinguishes the Asiatic from the other peoples. Table IV B indicates, indeed, that the Tibetan B type differs markedly from all the Asiatics (being larger than they are) for several of the long head measurements though it finds its counterpart in every case in one of the unallied races. This is notably so for head length (F), horizontal circumference (U), face height ($G'H$) and face breadth (J).

The races of Tables IV A and IV B have been arranged from left to right in order of their Coefficients of Racial Likeness with the Tibetans. No two of the horizontal arrays of indices, angles, or lengths are in the same, or in any very similar, order. Evidently none of these characters alone can give a measure of Racial Affinity even in the roughest way.

A comparison of the skull measurements has shown that the people of the southern provinces of Tibet—and though to a certain extent hybrid they are probably typical of the bulk of the population of the country—belong to the family of races which now inhabit Southern China, Burma, the Malay Peninsula and, most likely, the line of the Himalayas from Assam to the western border of Nepal. The races inhabiting this vast tract of country appear to be all more or less inter-related, although a more detailed study of their physical anthropology will, no doubt, disclose many sub-types among them. The Khams Tibetans do not appear to belong to this stock and it may not be unprofitable to compare them with other aboriginal peoples. The Moriori, Maori, Aino, Fuegians and Eskimo are races of man as

geographically remote from each other as any which could be named and it may be thought idle to seek for affinities between them. But they have this in common, that they are border-land peoples who have been pushed out to the far extremities of the great land masses of the world and having once arrived at their present habitats they were probably undisturbed by the great race upheavals which might exterminate or hybridise other races. Hence these primitive peoples appear to have preserved a more or less pure type since very early times. It may be that the history of the Khams Tibetans has been analogous to that of these other races; isolation in this case being due to the inaccessibility of Eastern Tibet. The importance of such a conception will be obvious to those who put their faith in types rather than in races of mankind.

For comparative purposes we are able to use the Moriori means given by Thomson* and the Maori means given by Scott†. The Fuegian data are more slender, but means are given for a number of characters in Thomson's paper (p. 98). The individual measurements were collected from various sources and reduced in the Biometric Laboratory‡.

It is probable that the Aino of Japan and the Tibetans of Khams are the modern representatives of two distinct aboriginal races of Eastern Asia, but their origins and relationships cannot yet be determined from a craniometric standpoint because there is no adequate data relating to the skulls of the inhabitants of Mongolia and the greater part of Siberia. The Eskimo to the east, however, have attracted more attention, their gr at importance in the study of ethnography has long been insisted on and consequently several collections of their skulls have been measured and described. We are in a position to determine whether there is any affinity between the Eskimo and either the Aino or the Tibetans.

Though several numerical descriptions of series of Eskimo skulls have been published, there is no single one which we can use for comparative purposes, since they deal with small numbers or give insufficient measurements. Consequently it became necessary to compile a table of means for male skulls from the data given by several craniometricians. Measurements were taken from the following sources:

(*a*) The German Anthropological Catalogues: there are six Eskimo in the Göttingen (1874) portion, one in the Freiburg (1878) and three in the Leipzig (1886) portion.

(*b*) Flower's catalogue of 18 specimens in the Museum of the Royal College of Surgeons (1879).

(*c*) Virchow, *Archiv für Anthropologie*, Bd. IV. (1870), p. 89 (three skulls).

(*d*) Duckworth, *Journal of the Anthropological Institute*, Vol. XXV. (1895), p. 73, and Vol. XXX. (1900) (24 skulls in all).

(*e*) Quatrefages and Hamy, *Les Crânes des Races Humaines* (1882), p. 440, and p. 441 (footnote (1)). Measurements of 11 skulls are given.

* E. Y. Thomson, "A Study of the Crania of the Moriori." *Biometrika*, Vol. XI. pp. 82—135.

† Scott's means for the Maori are quoted on p. 98 of Thomson's memoir.

‡ In place of some of the Fuegian means given on p. 98 of Thomson's paper which are based on only two skulls I have used the Hultkrantz means of six skulls (p. 100).

(*f*) Bessels, *Archiv für Anthropologie*, Bd. VIII. (1875), pp. 116—118* (68 skulls).

The descriptions of the measurements taken by some of these writers are very vague and incomplete, but none are included which had not been determined by methods the same as or very similar to ours.

All the skulls were said to be those of pure Eskimo, but since the sources consulted had been so various it became desirable to test in some way the homogeneity of the whole. This was done by dividing the material into two groups; an eastern one including all the Greenlanders and a western comprising the skulls from Labrador, Baffin Land and the Arctic Archipelago. No skull had been obtained from a locality west of Melville Island so that the Eskimo of Alaska were quite unrepresented. The mean characters of the two groups appeared to be very similar and the Coefficient of Racial Likeness between them was 1·66 ± ·11 for 19 characters and 0·21 ± ·21 for 5 indices. These low values showed, in my opinion, that the two groups might be combined to give a population which would be as homogeneous as the racial ones with which the craniologist usually has to deal. There can have been little if any European or Indian admixture in the western group—the large majority of the skulls having been obtained from the environs of Smith Sound—so it is probable that the skulls from Greenland represented a pure type also. A comparison of cephalic indices will illustrate this point in a way which will appear more satisfactory to craniologists who do not trust figures far. The following mean values of $100\,B/L$ were given for the Eskimo of Greenland by the various authorities cited: 72·2 (3)†, 72·0 (5), 71·8 (11), 71·6 (3), 70·8 (15), giving a weighted mean 71·4 (37). For Labrador and the islands to the west of Greenland the values are: 72·3 (3), 72·0 (4), 71·7 (11), 71·3 (67), 69·4 (4) and the weighted mean of these is 71·3 (89). These records appeared to be in excellent agreement for this and other characters, but a discordant series was found in a paper by Brierley and Parsons‡. This gives measurements of 13 supposed male Eskimo skulls which had been procured from the west coast of Greenland, and it is said that care was taken to include none that were not of the pure native type. One of these, however, has a cephalic index of 92·6 and the authors suggest that the skull was that of a European sailor. Excluding this one the mean index is 74·6 (12) and this can hardly be reconciled with the values given above. Since some of the other measurements

* Bessels gives measurements of 101 Eskimo skulls, but unfortunately they are not sexed, although he supposed that there were 28 females among them. The collection included 65 which had previously been measured and sexed (52 ♂ and 13 ♀) by Otis and a few of his means are quoted by Quatrefages and Hamy (*op. cit.* p. 441). I have not been able to see the original paper by Otis. By considering three characters—length, breadth and capacity—and supposing that the distribution of Bessels' 101 individuals could be represented by two normal and partly overlapping curves, the means of which were the values given by Otis, I was able, by approximate methods, to pick out 33 skulls which with great probability were all female. The means of the remaining 68 were calculated, supposing them to be all males.

† The numbers in brackets are of the number of skulls on which the means were based.

‡ *Journal of the Anthropological Institute*, Vol. XXXVI. (1906). The table of measurements is given on p. 118. It may be noticed that these skulls were collected some years after the majority of the others we have considered.

were also not in agreement with the general type, we should not feel justified in including the means of Brierley and Parsons with the others. The Eskimo skull differs markedly from those of all our Asiatic races in having a very low cephalic index and in being greater in basio-bregmatic height than in breadth.

The mean measurements for the Khams Tibetans and the various primitive peoples we have mentioned are given in the following table and the Coefficients of Racial Likeness in Table VII. The indices in square brackets are the ratios of mean measurements and not the means of the individual indices. In the case of the Aino and Eskimo it is not stated which orbit was used, but for the other

TABLE VI.

Comparative Table of Means (Male) of Primitive Races.

Character	Tibetan B	Aino	Eskimo	Fuegians	Moriori	Maori
L	185·5 (14)	185·8 (88)	188·2 (148)	192·0 (34)	186·9 (35)	185·5 (43)
B	139·4 (14)	141·2 (88)	134·1 (146)	145·0 (34)	141·4 (34)	140·1 (43)
B'	94·3 (15)	96·2 (88)	94·9 (20)	97·1 (34)	95·3 (35)	95·7 (43)
OH	115·5 (15)	119·3 (88)	—	—	117·1 (34)	—
H'	134·1 (15)	138·7 (88)	140·0 (56)	141·0 (34)	135·9 (34)	137·6 (43)
LB	99·2 (15)	105·4 (88)	104·9 (39)	106·0 (6)	105·6 (34)	103·9 (43)
Q	312·1 (15)	328·5 (84)	306·0 (3)	316·1 (6)	316·1 (34)	307·5 (43)
S	378·6 (14)	372·8 (77)	373·7 (29)	381 (6)	368·8 (34)	378·1 (43)
S_1	129·3 (15)	—	127·2 (26)	—	126·0 (34)	—
S_2	126·1 (14)	—	126·5 (26)	—	112·4 (34)	—
S_3	122·7 (15)	—	119·4 (26)	—	121·1 (34)	—
U	525·6 (14)	522·5 (88)	523·4 (145)	531 (34)	522·8 (34)	518·0 (43)
G'H	76·5 (15)	69·8 (73)	72·4 (25)	—	76·4 (35)	71·4 (43)
J	137·5 (15)	137·3 (74)	136·4 (101)	143·6 (34)	137·4 (34)	136·8 (43)
NH, *L*	54·9 (15)	50·5 (79)	53·5 (33)	54·6 (6)	57·3 (35)	53·8 (43)
NB	27·1 (15)	25·6 (78)	23·5 (50)	25·3 (6)	25·3 (35)	25·6 (43)
$O_1 L$	43·4 (15)	40·9 (79)	39·8 (26)	40·0 (6)	44·4 (35)	40·5 (43)
$O_2 L$	36·7 (15)	34·9 (79)	35·5 (35)	35·3 (6)	37·3 (34)	35·0 (43)
G_1	51·5 (13)	53·0 (75)	—	—	53·1 (33)	52·4 (43)
G_1'	47·7 (13)	—	47·0 (3)	—	50·1 (34)	—
G_2	43·6 (14)	38·2 (74)	40·3 (3)	—	38·9 (34)	37·3 (43)
GL	97·2 (15)	104·9 (69)	104·2 (23)	106·0 (6)	100·9 (34)	100·8 (43)
GB	100·7 (15)	102·1 (76)	100·4 (29)	110·0 (34) (?)	101·2 (34)	—
fml	37·4 (15)	—	38·1 (14)	35·3 (6)	35·3 (33)	35·2 (43)
fmb	31·4 (15)	—	29·1 (14)	31·0 (6)	30·8 (34)	30·8 (43)
C	1537·7 (14)	1462 (76)	1472·2 (109)	1474 (34)	1422·1 (28)	1476 (43)
DC	22·7 (15)	—	21·8 (14)	—	—	—
100 *B*/*L*	75·3 (14)	76·5 (88)	[71·4] (145)	76·6 (34)	76·1 (34)	75·4 (43)
100 *H'*/*L*	72·1 (14)	74·7 (88)	[74·2] (55)	73·4 (34)	72·8 (34)	74·7 (43)
100 *B*/*H'*	104·5 (14)	101·0 (88)	[95·9] (40)	[102·8] (34)	104·5 (34)	[101·8] (43)
100 *NB*/*NH*	49·5 (15)	50·7 (78)	[42·6] (23)	—	43·9 (34)	47·9 (43)
100 O_2/O_1, *L*	84·6 (15)	85·3 (80)	[89·4] (31)	87·4 (6)	84·0 (33)	86·1 (43)
100 G_2/G_1	84·1 (12)	72·1 (72)	•—	—	73·2 (34)	[71·2] (43)
100 *G'H*/*GB*	76·0 (15)	68·4 (65)	74·0 (3)	—	74·1 (34)	—
100 *fmb*/*fml*	84·1 (15)	—	[76·4] (14)	—	87·4 (34)	—
P∠	85°·7 (14)	82°·0 (67)	—	83°·6 (34)	84°·7 (34)	—
N∠	65°·5 (15)	—	—	—	64°·8 (34)	—
A∠	68°·8 (15)	—	—	—	71°·8 (34)	—

races the measurements given are of the left orbit. In most cases the nasal heights were measured on the left side.

TABLE VII.

Coefficients of Racial Likeness. All Male *.*

		Tibetan B	Aino	Eskimo	Fuegians	Moriori	Maori
Tibetan B	All Characters	—	13·61 ± ·10 (23)	12·28 ± ·09 (26)	4·97 ± ·10 (21)	4·85 ± ·09 (30)	8·14 ± ·09 (26)
	Indices and Angles only	—	11·94 ± ·17 (8)	18·58 ± ·18 (7)	1·17 ± ·21 (5)	4·74 ± ·14 (11)	6·55 ± ·18 (7)
Aino	All Characters	13·61 ± ·10 (23)	—	26·30 ± ·10 (22)	6·69 ± ·11 (19)	21·80 ± ·10 (25)	8·44 ± ·10 (23)
	Indices and Angles only	11·94 ± ·17 (8)	—	63·73 ± ·19 (6)	3·72 ± ·21 (5)	17·18 ± ·17 (8)	1·38 ± ·19 (6)
Eskimo	All Characters	12·28 ± ·09 (26)	26·30 ± ·10 (22)	—	19·53 ± ·10 (21)	25·06 ± ·10 (24)	13·25 ± ·10 (24)
	Indices and Angles only	18·58 ± ·18 (7)	63·73 ± ·19 (6)	—	34·93 ± ·21 (5)	50·72 ± ·19 (6)	31·80 ± ·19 (6)
Fuegians	All Characters	4·97 ± ·10 (21)	6·69 ± ·11 (19)	19·53 ± ·10 (21)	—	4·56 ± ·10 (21)	5·67 ± ·11 (20)
	Indices and Angles only	1·17 ± ·21 (5)	3·72 ± ·21 (5)	34·93 ± ·21 (5)	—	0·96 ± ·21 (5)	1·14 ± ·24 (4)
Moriori	All Characters	4·85 ± ·09 (30)	21·80 ± ·10 (25)	25·06 ± ·10 (24)	4·56 ± ·10 (21)	—	8·48 ± ·09 (26)
	Indices and Angles only	4·74 ± ·14 (11)	17·18 ± ·17 (8)	50·72 ± ·19 (6)	0·96 ± ·21 (5)	—	5·33 ± ·18 (7)
Maori	All Characters	8·14 ± ·09 (26)	8·44 ± ·10 (23)	13·25 ± ·10 (24)	5·67 ± ·11 (20)	8·48 ± ·09 (26)	—
	Indices and Angles only	6·55 ± ·18 (7)	1·38 ± ·19 (6)	31·80 ± ·19 (6)	1·14 ± ·24 (4)	5·33 ± ·18 (7)	—

This table shows clearly that the Eskimo skull stands quite apart from all the others and has no affinity with the primitive oriental types. But the inter-relations of the other races considered are of greater interest.

The most significant differences between the Tibetan B and Moriori types are in height and breadth of nose and breadth of palate, these giving very divergent values for the indices NB/NH and G_2/G_1. The calvariae are very similar except that the Tibetan has the shorter base (LB) and the nose is the only portion of the face which distinguishes one from the other. The Maori type differs more markedly from the Tibetan and particularly in its lesser breadths of palate and orbit and smaller face height. But the calvariae again are very similar for all measurements except LB. The Fuegian skull is no further removed from the Tibetan than the Moriori was found to be, but it does not diverge in the same way from it. The greater size of the Fuegian calvaria is the character which distinguishes it; the shape of the two, as judged by indices, is very similar.

If the Coefficients of Racial Likeness for all characters given in Table VII are compared with those of Table II it will be seen that the Tibetan B type resembles the Moriori and Fuegian types as closely as it does any Asiatic type with the exception of the Karens and that was deduced from only 8 skulls so that little reliance

* The figures in round brackets give the number of characters on which the coefficients are based.

can be placed on its accuracy. But none of these relationships are as close as those which were found between the Tibetan A and neighbouring races. It would be inadvisable to lay stress on these apparent racial affinities while our data are so meagre, but perhaps we are warranted in saying now that craniometric evidence may necessitate theories of very extensive prehistoric wanderings of human types.

The inter-relations of the three southern races, for which the Coefficients of Racial Likeness are given in Table VII, appear to be of peculiar interest. It is surprising to find that both the Maori and the Moriori resemble the Fuegians more than they do each other, and it becomes more than ever difficult to believe "that there has not at some time been a race-link between the present denizens of South America and the denizens of extreme Polynesia*." It is a very noteworthy fact that the South American skull differs most significantly from each of the Polynesian ones in very much the same way; the most aberrant characters in the case of the Maori are L, B, J and U and L, B, J and O_1 for the Moriori. If the Moriori and Maori mean measurements are compared (Table VI) it is, at first, not at all evident why the Coefficient of Racial Likeness between them should be so large as it is. The longer lengths and arcs, with the possible exception of Q, and their indices appear to correspond very closely. The characters which are responsible for raising the coefficient are O_1L, O_2L, $NB/NH\,(R)$ and $NH\,(R)$; no others are of any importance. The most influential of these is O_1L, and, if the three orbital measurements (O_1L, O_2L, $O_2/O_1\,(L)$) are not included, the coefficient is reduced from 8·48 to 3·91: this value is of the order we might have anticipated, but it is still high. If nasal measurements are also left out of account, the coefficient, now based on 20 characters, becomes 2·23, which indicates a close relationship†. This emphasises again the importance of obtaining a criterion of racial affinity which takes into account not one or even 10 characters alone, but all the chief arcs, lengths, indices and angles of the skull.

Turning to the Aino, it is surprising to find that the coefficients with the Fuegians and Maori are so small, these denoting closer relationships than that between the Ainos and Tibetans. The low values of the coefficients between the Fuegians and all the other races with the exception of the Eskimo are the really noteworthy feature of Table VII. The lowering of the values when indices and angles only are considered is very significant. With the Aino the most divergent characters are H/L, L, B, J and NH. The very low values for the coefficient for the Maori and Aino when indices and angles only are considered appear to indicate that the two types are very similar in shape, but it is unfortunate that no more than 6 characters are available for comparison. The coefficient for all characters is so much larger than the other owing to the great difference in length between the transverse arcs (Q). If that is left out of account the coefficient becomes 3·04 for 22 characters. The difference between the nasal heights is also very significant

* E. Y. Thomson, "A Study of the Crania of the Moriori." *Biometrika*, Vol. XI. p. 101.

† With regard to this point it must, however, be noted that personal equation is likely to be of much importance in orbital length and nasal height, and there is some evidence that Scott and Thomson did not use the same conventions. See *Biometrika*, Vol. XI. pp. 92—94 and Vol. XIII. p. 345.

and if the three nasal measurements (NH, R, NB and NH/NB) are neglected as well as Q we find C. L. = 1·01 for 19 characters. Without more abundant evidence it would be rash to assign any ethnographic meaning to these results. After noting such a suggestive resemblance it is remarkable to find that the Moriori and Aino are as unlike each other as any two races we have compared in this paper. The Maori it would thus seem stand in an intermediate position between the Moriori and Aino types.

It has frequently been suggested that one or other single measurement of the skull can by itself be used as a racial criterion. While it is now generally recognised that no direct measurement can fulfil this purpose, yet there are certain indices which still are used to support the claim. Are the differences between the mean racial values of these indices at all correlated with the corresponding Coefficients of Racial Likeness? The two methods of measuring relationship are compared in the following tables, which deal with four of the most important indices suggested.

TABLE VIII.

Comparison of Mean Cephalic Indices with C. L.'s. All Males.

Race compared	Values of 100 B/L	With Tibetan A		With Tibetan B	
		Difference of 100 B/L	C.L.	Difference of 100 B/L	C.L.
Burmese A (Burmans)...	82·9	+3·1	5·71	+7·6	10·96
Malayan	81·7	+1·9	2·88	+6·4	11·17
Burmese B (Hybrids?) ...	80·4	+0·6	0·80	+5·1	4·83
French	79·8	0·0	9·12	+4·5	22·96
Tibetan A	79·8	—	—	+4·5	6·46
Burmese C (Karens?) ...	79·5	−0·3	2·36	+4·2	3·46
Chinese	78·9	−0·6	3·23	+3·6	6·37
Congo Negroes	78·0	−1·8	7·23	+2·7	21·61
Fuegians	76·6	−3·2	18·86	+1·3	4·97
Aino	76·5	−3·3	13·82	+1·2	13·61
Moriori	76·1	−3·7	12·78	+0·8	4·85
Hindu	75·8	−4·0	6·56	+0·5	18·34
Maori	75·4	−4·4	10·55	+0·1	8·14
Tibetan B	75·3	−4·5	6·46	—	—
Maravar	74·6	−5·2	12·34	−0·7	25·72
Whitechapel English ...	74·3	−5·5	14·59	−1·0	11·93
Naqada Egyptians ...	73·0	−6·8	17·64	−2·3	18·32
Eskimo	71·4	−8·4	24·15	−3·9	12·28

The simotic 100 SS/SC and mesodacryal 100 DS/DC indices are tabulated below in the same way, the values for the Whitechapel English, Congo and Aino being taken from the paper on the nasal bridge by Miss K. V. Ryley and Dr Julia Bell*.

Whether we confine our attention to Asiatic races or consider all given in these three tables, it is clear that there is no close relation between the Coefficients of Racial Likeness and the values of the single indices. The orders in which these criteria would arrange the races appear to have no obvious points of similarity and we must accept the conclusion that none of these indicial characters can be used alone to

* *Biometrika*, Vol. IX. p. 404.

TABLE IX.

Comparison of Mean Racial Values of the Conjoint Index† $100\,(B-H')/L$ *with C. L.'s. All Males*‡.

Race compared	Values of $100\,(B-H')/L$	With Tibetan A		With Tibetan B	
		Difference of $100\,(B-H')/L$	C. L.	Difference of $100\,(B-H')/L$	C. L.
French	+7·2* (56)	+2·5	9·12	+3·7	22·96
Tibetan A	+4·7 (17)	—	—	+1·2	6·46
Burmese A (Burmans)	+4·5* (43)	−0·2	5·71	+1·0	10·96
Whitechapel English	+4·2 (120)	−0·5	14·59	+0·7	11·93
Moriori	+3·5 (33)	−1·2	12·78	0·0	4·93
Tibetan B	+3·5 (14)	−1·2	6·46	—	—
Fuegians	+3·2*	−1·5	18·86	−0·3	4·97
Malayan	+3·2 (76)	−1·5	2·88	−0·3	11·17
Burmese B (Hybrids?)	+2·9* (7)	−1·8	0·80	−0·6	4·83
Congo Negroes ...	+2·6* (48)	−2·1	7·23	−0·9	21·61
Chinese	+1·3 (69)	−3·4	3·23	−2·2	6·37
Aino	+0·9* (88)	−3·8	13·82	−2·6	13·61
Burmese C (Karens?)	+0·8* (8)	−3·9	2·36	−2·7	3·46
Maori	+0·7*	−4·0	10·55	−2·8	8·14
Hindu	−0·2 (44)	−4·9	6·56	−3·7	18·34
Naqada	−0·3* (130)	−5·0	17·64	−3·8	18·32
Maravar	−0·6* (38)	−5·3	12·34	−4·1	25·72
Eskimo	−2·8*	−7·5	24·15	−6·3	12·28

* Indices marked with an asterisk were calculated from the mean values of $100\,B/L$ and $100\,H'/L$ and not from the individual measurements.

TABLE X.

Comparison of Mean Racial Values of 100 *SS/SC with C. L.'s. All Males.*

Race compared	100 *SS/SC*	With Tibetan A		With Tibetan B	
		Difference of 100 *SS/SC*	C. L.	Difference of 100 *SS/SC*	C. L.
Congo Negroes	26 (?)	− 6	7·23	− 9	21·61
Tibetan A	32 (16)	—	—	− 3	6·46
Burmese A (Burmans)...	33 (41)	+ 1	5·71	− 2	10·96
Tibetan B	35 (15)	+ 3	6·46	—	—
Aino	43 (?)	+11	13·82	+ 8	13·61
Burmese B (Hybrids?)...	45 (8)	+13	0·80	+10	4·83
Whitechapel English ...	51 (?)	+19	14·59	+16	11·93
Burmese C (Karens?) ...	54 (7)	+22	2·36	+19	3·46

indicate racial affinity, except perhaps in isolated cases, and we do not believe that any single direct or indicial measurement can perform that function. Apart from

† Professor Karl Pearson has suggested that the index $100\,(B-H')/L$ should be known by this name.

‡ The values of the index are taken from *Biometrika*, Vol. XIII. p. 245. The tables there give, except for the Burmese types, the values of $100\,(B-H')/L$ and not $100\,(B-H)/L$ as stated. We are obliged to use the basio-bregmatic height (H') because the more important vertical height (H) is as yet available for only a few races. It must be carefully noted that our H' is the H of Thomson, Benington and Macdonell.

that, of course, the measurements of isolated characters are still of great intrinsic interest. If the Coefficients of Racial Likeness calculated from indices and angles only are compared with these single indicial characters in the same way, we are forced to accept the same conclusion with regard to the latter.

TABLE XI.

Comparison of Mean Racial Values of 100 *DS/DC with C. L.'s. All Males.*

Race compared	100 *DS/DC*	With Tibetan A		With Tibetan B	
		Difference of 100 *DS/DC*	C. L.	Difference of 100 *DS/DC*	C. L.
Tibetan A	37 (16)	—	—	− 2	6·46
Tibetan B	39 (15)	+ 2	6·46	—	—
Congo Negroes	39 (?)	+ 2	7·23	0	21·61
Burmese A (Burmans)...	46 (42)	+ 9	5·71	+ 7	10·96
Burmese B (Hybrids?)...	50 (8)	+13	0·80	+11	4·83
Aino	50 (?)	+13	13·82	+11	13·61
Whitechapel English ...	57 (?)	+20	14·59	+18	11·93
Burmese C (Karens?) ...	62 (7)	+25	2·36	+23	3·46

8. *Study of the Contours of the Tibetan Crania.*

Drawings of the Transverse, Horizontal and Sagittal contours were made with the Klaatsch contour tracer in the manner described by Benington in his paper on Cranial Type contours* and the method of constructing the type from them described there was used. The mean measurements of the individual contours determine the points of the type contour.

(*a*) *The Transverse Vertical or Auricular Coronal Section.* This section is drawn from one auricular point† to the other through the apex (*A*), the latter point being defined as the highest point of the section perpendicular to the Frankfurt horizontal and passing through the auricular points. The vertical axis *MA* is the perpendicular bisector of the line joining the auricular points—the "auricular line." Lines parallel to the auricular line are drawn dividing *MA* into ten equal parts. These lines are numbered upwards from 1 to 10 and measurements are taken to left and right of the vertical axis. Another parallel $A\frac{1}{4}$ is drawn at a distance of one quarter of the last section (*A* 10) from the apex, and yet another ($M\frac{1}{4}$) the same distance from the "auricular line." The points where the zygomatic ridges are crossed (*ZR*. *R* and *ZR*. *L*) are marked on the individual contours and located from *M* by rectangular coordinates, *x* (along *MA*) and *y*. The upper trough of the ear passage beyond the auricular points was found in most cases to slope forwards towards the face so that little of it could be traced and no attempt has been made to represent it on the type contour.

The mean measurements of the individual section from which the type was

* *Biometrika*, Vol. XI. p. 129.

† These are not the "auricular points by definition" but those on which the skull rested when in the craniophor. See p. 201.

constructed are given in the following table and the types themselves are Figs. 1 and 2.

TABLE XII.

Tibetan Transverse Vertical Contours. Mean Values.

	MA	*IR=IL*	*M¼R*	*M¼L*	*2R*	*2L*	*3R*	*3L*	*4R*	*4L*	*5R*	*5L*	*6R*	*6L*
Tibetan A (17)	113·9	59·6	62·0	62·4	63·2	64·1	66·1	67·0	68·3	69·3	68·4	69·8	67·3	68·6
Tibetan B (15)	115·9	61·6	64·2	64·3	65·6	65·7	67·5	67·6	69·0	69·0	69·2	68·8	68·0	67·4

	7R	*7L*	*8R*	*8L*	*9R*	*9L*	*10R*	*10L*	*A¼R*	*A¼L*	*ZR.R*		*ZR.L*	
											y	*x*	*y*	*x*
Tibetan A (17)	65·0	66·4	60·3	61·8	51·3	52·9	36·5	37·0	19·1	17·8	62·4	3·2	62·6	3·1
Tibetan B (15)	65·3	64·5	60·0	59·3	50·7	50·1	36·6	35·1	20·2	17·9	64·8	3·9	64·9	4·1

Drawings of cranial contours were originally proposed for the purpose of comparing and measuring crania individually and thus obtaining mean racial characters which could not be easily found otherwise, but the use afterwards made of them in constructing average or type contours has become of far greater interest than the original purpose. The comparative material available is unfortunately still meagre. It consists of the Congo, Egyptian, Eskimo, Guanche and English sections of Benington*, the Moriori of Thomson† and the recent Burmese data of Tildesley‡. The three sections we have taken are given for all these and tracings accompany the papers. We are only considering male crania and for these the available types are:

I. Congo (Gaboon) Fernand Vaz, 1864 ... ... 50♂ Crania
II. Congo „ „ „ 1880 ... ... 20♂ „
III. Congo (Batetelu race) ... ... 41♂ „
IV. Egyptian (B.C. 600–200) ... ... 100♂ „
V. Eskimo ... ... 31♂ „
VI. Guanche ... ... 14♂ „
VII. Seventeenth Century English (Whitechapel) ... 100♂ „
VIII. Moriori ... ... 33♂ „
IX. Burmese { Type A (Burmans) ... ... 44♂ „ / Type B (Supposed Hybrids) ... ... 7♂ „ / Type C (Supposed Karens) ... ... 8♂ „

and to these we are now able to add

X. Tibetan { Type A (Sikkim region)... ... ... 17♂ „ / Type B (Khams)... ... ... 15♂ „

* *Biometrika*, Vol. VIII. pp. 155—195.
† *Ibid.* Vol. XI. pp. 113—123.
‡ *Ibid.* Vol. XIII. pp. 188—208.

Many of these series are very short and we must be particularly careful to avoid the omnipresent danger of drawing conclusions without paying due regard to the divergencies which would lie within the limits of probable error. Diagrams

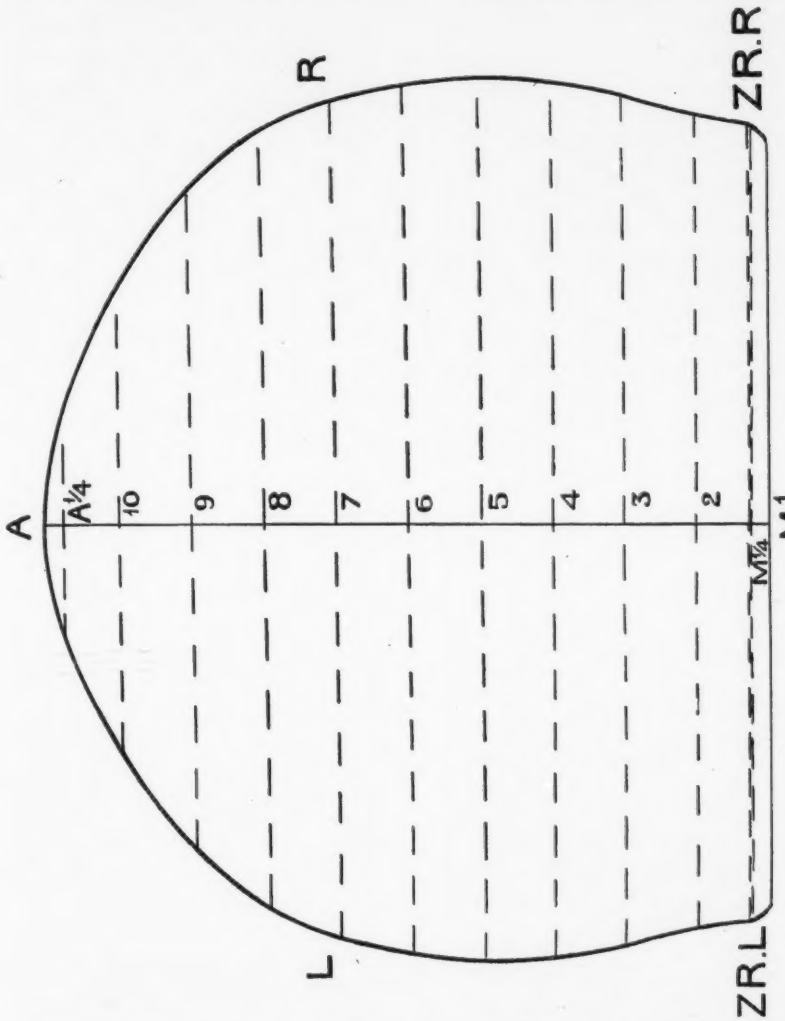


Fig. 1. Tibetan A. ♂. Vertical Type Contour. (17 Crania.)

indicating the range of variation of the contour lines have been given by Benington* for the seventeenth century English series above, the largest we are

* *Biometrika*, Vol. VIII. pp. 143, 145, 147 for the transverse, horizontal and sagittal contours respectively.

dealing with. Points were plotted at distances of twice the probable error of the mean ordinate from each mean ordinate and on either side of it and the outer ring and inner ring of these were splined up to give two diagrams. The outer one is

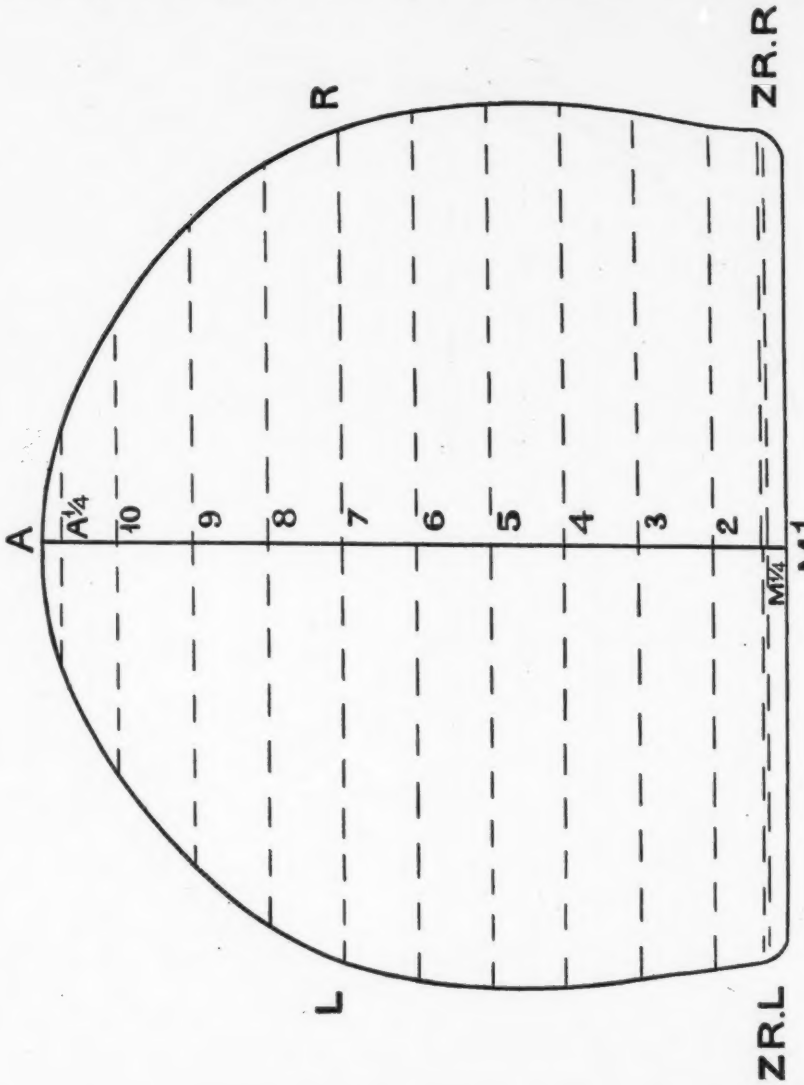


Fig. 2. Tibetan B. ♂. Vertical Type Contour. (15 Crania.)

rather larger than the type and the inner one rather less than it. The space between was termed the type zone and of this it was said, "If two type zones for

two races do not fall wholly or at definite points well outside each other, we suggest that the races represented by these types are either undifferentiated or closely related*."

Now the transverse vertical type zone shown for the English crania is 1·4 mm. wide at the apex and it tapers down to 0·8 mm. on nearing the auricular points. Assuming that the standard deviations of English and Tibetan skulls are approximately the same, we calculate, as Tildesley has done for the Burmese series†, that the Tibetan A type zone should have a width of 3·4 mm. at the apex and 1·9 mm. on nearing the base line; the corresponding measures for the Tibetan B series will be 3·6 mm. and 2·1 mm.‡

Not all the transverse vertical contours given for the above races (p. 228) were prepared in precisely the same way. Those of Dr Benington, which we have numbered I to VII, were said to have been drawn from auricular point to auricular point, but the types fall straight down to the auricular line without any sudden bend inwards, such as might have been expected, and this gives them the appearance of terminating at the zygomatic ridges. Neither of the ordinates $M\frac{1}{4}$ or the points $ZR(R)$ and $ZR(L)$ was marked and there was thus no detailed description of the region immediately above the auricular points. These latter appear to have been splined up with the extremities of the parallels 2, 3, 4, etc. I was able to confirm this supposition by examining the individual contours on which the types had been based. In many of them the curve round and below the zygomatic ridge was quite clearly shown and further evidence was afforded by comparing the *OH* measurements of the transverse and sagittal§ contours of the same skull. These were found to be in close agreement in every case. When comparing the transverse type contours of the Congo, Egyptian, Eskimo, Guanche and English crania then, it must be remembered that they should in reality widen rapidly for a short distance above the auricular points and then rise more vertically than shown to pass through the extremities of the second parallels.

I was also able, through the kindness of Professor Karl Pearson, to examine the individual contours of the Moriori crania drawn by Thomson and they had apparently been made in a somewhat different way. In not a single one of them was there any curve inwards towards the so-called auricular line. In most, indeed, the contour widened at the base in a way that must have been racially peculiar to the Moriori. There can be little doubt that the terminals used were the zygomatic ridges and not the auricular points. This view was confirmed by finding that the height of the transverse type contour was 5·5 mm. less than the auricular height given by the sagittal type‖.

* *Biometrika*, Vol. VIII. p. 129. This paper describing Dr Benington's work was prepared for the press by Professor Karl Pearson.

† *Biometrika*, Vol. XIII. p. 226.

‡ The type zones were marked on the tracings of the Tibetan type contours which I used for comparative purposes, but they have not been reproduced because they would confuse the already crowded drawings.

§ Though not reproduced in the types, the auricular points had been marked on Dr Benington's individual sagittal contours.

‖ Comparing Figs. I and III, *Biometrika*, Vol. XI. pp. 113 and 115 respectively.

We have used precisely the method described by Miss Tildesley in her study of the Burmese skull; she was the first to measure the ordinates $M\frac{1}{4}$ and to mark the positions where the zygomatic ridges are crossed.

The right and left of the individual contours were interchanged in constructing the type so the *norma occipitalis* is shown. We have followed Thomson and Tildesley in this: Benington's contours show the *norma facialis*. The right and left of Table XII must be interchanged to correspond with the diagrams.

If the auricular lines are superposed the two Tibetan types appear very similar. Between parallels 4 and 9 the divergence between them is less than 0·1 mm. and the larger B type only falls outside the type zone of A a short distance in the region of the apex and immediately above the two auricular points. The greater width here, which is especially marked between the points $ZR(R)$ and $ZR(L)$, is quite significant.

The Burmese sections A and B are rather greater in height and very slightly less in breadth at the base than the Tibetan A one. We must hesitate to draw any conclusion from the much more marked narrowness of the lower half of the C contour, because that one represents such very slender data. In height the Tibetan B type is almost exactly equal to the three Burmese, but the greater width of the former between the zygomatic ridges, and of the lower half of the section generally, clearly distinguishes it from them. The curvatures of the vaults, compared by superposing the lines AM and the points A, also differ significantly; the Tibetan B falling away more rapidly from the apex. The Tibetan A section had the more rounded form in common with the Burmese and we can confidently say that these two A types resemble each other more than either does the Tibetan B.

If the line joining the zygomatic ridges of the Tibetan A figure is placed on top of that joining the points marked L. A., R. A., as we suppose they represent the same length, of the Moriori contour the correspondence between the two sections is seen to be remarkably close. The greatest divergence between them for their whole length is 0·8 mm. so that, if nothing was known of their origin, the probability of their representing random samples from the same population would certainly be very high. This correspondence with the Moriori is certainly much closer than that between the Tibetan A and any of the Burmese types. The Tibetan B contour, of course, differs from the Moriori in very much the same way that it differed from the Tibetan A.

The heights and the curves of the vaults of the three Congo series are very similar to those of the Tibetan A type, but the latter differs very appreciably from them in being broader in the middle and for the lower half; the divergence increases as the auricular line is approached. The auricular breadth of the Tibetan is only equalled by that of the Guanche and English and is exceeded by the Eskimo drawings. The English is the only one of these types to which it bears any striking resemblance; in that case the contour falls entirely within the type zone, though the correspondence is not nearly so close as with the Moriori. The vaults of all these

are of a smooth and rounded form and very much alike, except the characteristic one of the Eskimo which is more pointed than the others.

To the Tibetan B we fail to find any counterpart, for its great width at the second, third and fourth parallels distinguishes it from all and in the extreme cases, such as the Congo and Egyptian, it exceeds the other types by as much as 5 mm. It is true that the Eskimo has a greater auricular width than the Tibetan, but immediately above the auricular line the latter widens more rapidly.

When it is said, after comparisons have been made between type contours, that two races appear to be closely related or akin we are speaking only of relationship of form and do not for a moment mean to suggest that any such evidence can provide a criterion of remote consanguinity. Indeed, it will be shown later that two races almost as remote as any imaginable, may have very similar sections*. Even when all possible points have been marked on the sagittal contours there is no indication at all of the orbits, or of the nasal, palatine and foraminal breadths.

The important measurements which can be read off from the transverse contours are the interauricular length and the auricular height. A table of these and the resulting indices has been given by Thomson† and to that we are now able to add the measurements of the three Burmese type contours and the present Tibetan ones. Unfortunately the Moriori lengths cannot be included since the contours were not drawn in the same way as the others.

TABLE XIII.

Measurements on the Transverse Type Section. Males.

Race	Interauricular Length	Auricular Height	Index
Eskimo	128·0	114·2	89·2
English	122·4	111·5	91·1
Guanche	119·6	109·8	91·8
Tibetan B	123·2	115·9	93·3
Egyptian	117·4	110·2	93·9
Tibetan A	119·2	113·9	95·6
Congo, Bantu‡	116·8	113·6	97·3
Burmese A	118·2	116·5	98·6
Burmese C	114·6	115·6	100·9
Burmese B	114·6	115·7	101·0
Cro-Magnon	124·0	121·0	97·6

This table shows again that the Tibetan A resembles in this index the three Burmese types, while the Tibetan B differs from all those but is very similar to the English and Guanche.

* See the comparison between Whitechapel English and Tibetan B on p. 248.

† *Biometrika*, Vol. XI. p. 103.

‡ Fernand Vaz, 1864 Series.

(*b*) *The Glabella Horizontal Section.* This section is drawn through the glabella with the inverted skull with the Frankfurt horizontal orientation. The points fixing this are the nasion, a point previously marked above the left auricular passage, and the "gamma" in the occipital region. *F* and *O* being points marked on the contours immediately above the nasion and below the lambda respectively, lines dividing *FO* into ten equal parts and perpendicular to it are measured to left and right of that base line. These are numbered from 2, that being the first tenth division from *F*, to 10, the nearest to *O*. Two more parallels, $F1\frac{1}{4}$ and $F1\frac{1}{2}$, are taken $\frac{1}{4}$ and $\frac{1}{2}$ the distance *F*2 from *F* with the object of determining the frontal curvature more precisely and another, $O1\frac{1}{4}$ being $\frac{1}{4}$ of the same distance from *O*, serves the same purpose in the occipital region. The points where the temporal lines are crossed, *T* (*R*) and *T* (*L*), having been marked on the contours are orientated from *F* by rectangular coordinates, *T* (*x*) along *FO* and *T* (*y*).

The mean measurements from which the type contours were constructed (Figs. 3 and 4) are given below.

TABLE XIV.

Tibetan Horizontal Contours. Mean Values.

	FO	$F1\frac{1}{4}R$	$F1\frac{1}{4}L$	$F1\frac{1}{2}R$	$F1\frac{1}{2}L$	2*R*	2*L*	3*R*	3*L*	4*R*	4*L*	5*R*	5*L*	6*R*	6*L*
Tibetan A (17)	173·3	27·0	24·0	37·7	36·1	47·7	46·0	51·5	50·9	57·7	57·4	64·4	65·8	68·1	69·9
Tibetan B (15)	183·2	27·6	26·3	39·8	38·7	48·9	48·0	52·6	51·9	59·0	59·2	65·0	66·6	67·7	70·7

	7*R*	7*L*	8*R*	8*L*	9*R*	9*L*	10*R*	10*L*	$O1\frac{1}{4}R$	$O1\frac{1}{4}L$	*T*(*R*)*x*	*T*(*L*)*x*	*T*(*R*)*y*	*T*(*L*)*y*
Tibetan A (17)	68·7	70·5	65·4	67·2	57·7	59·7	44·2	46·5	24·5	25·4	17·6	18·2	48·4	46·7
Tibetan B (15)	68·1	70·6	64·6	67·2	57·2	59·9	44·2	46·6	24·5	24·5	16·9	18·1	49·1	48·6

The type contours of Benington do not show the parallel $F1\frac{1}{2}$ or mark the positions where the temporal lines are crossed; these were added by Thomson and have been included by subsequent workers in the Biometric Laboratory. Thus the drawings of the Moriori, Burmese and Tibetan crania have a more detailed and less rounded appearance in the frontal region than the others, though the additional points only make very slight differences to the path of the contour which would have been chosen without their guidance. The *norma verticalis* aspect is shown for all the types, but the right and left of the table above are those of the individual contours taken with the skull placed apex downwards and they must be reversed to correspond with the type contour.

The horizontal type zone given by Benington for 100 English crania has an approximate width of 1·2 mm. all round. The corresponding width for 17 crania will be 2·9 mm., and for 15, 3·1 mm.

In superposing horizontal contours the axis *FO* is used as the common base line and one or other of the points *F* and *O* should also be made coincident.

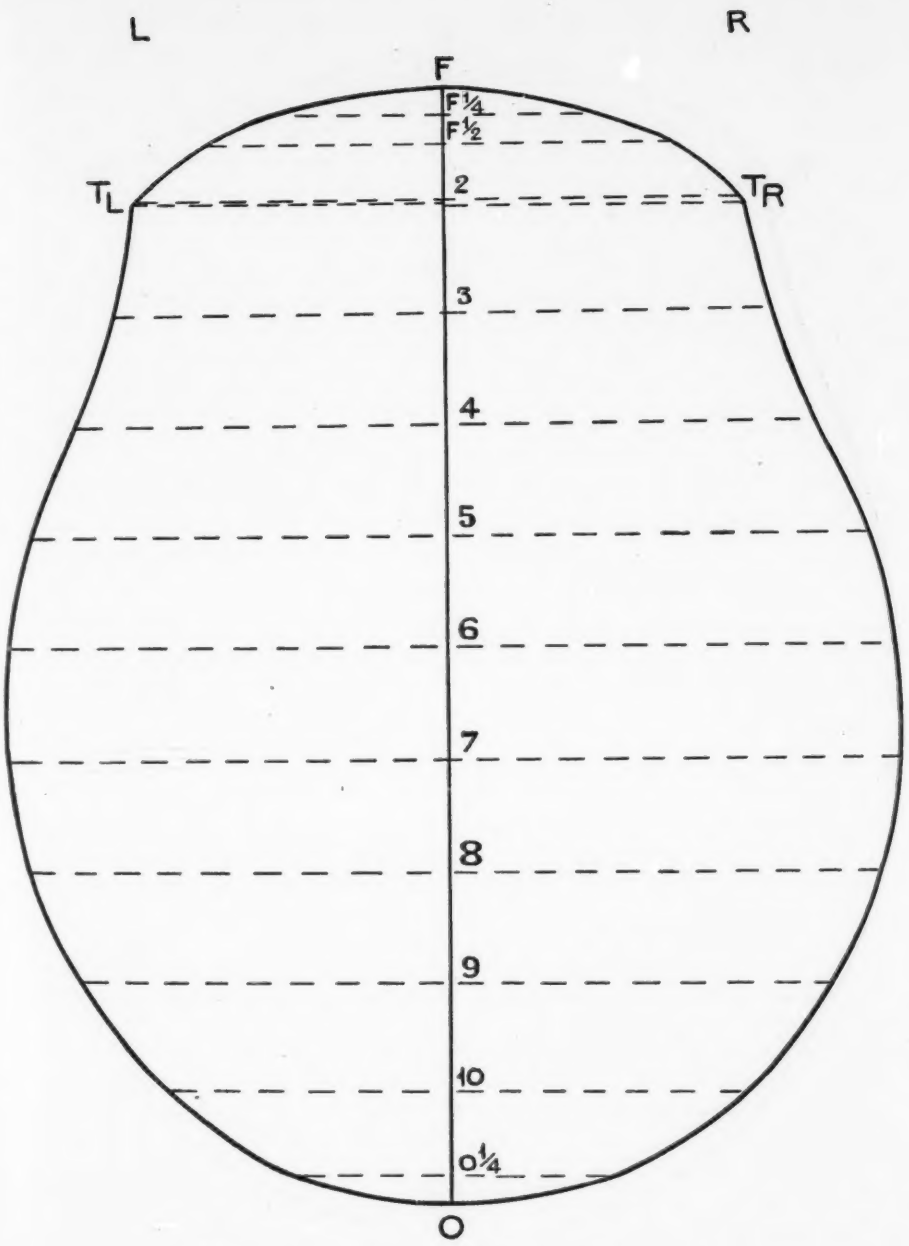


Fig 3. Tibetan A. ♂. Horizontal Type Contour. (17 Crania.)

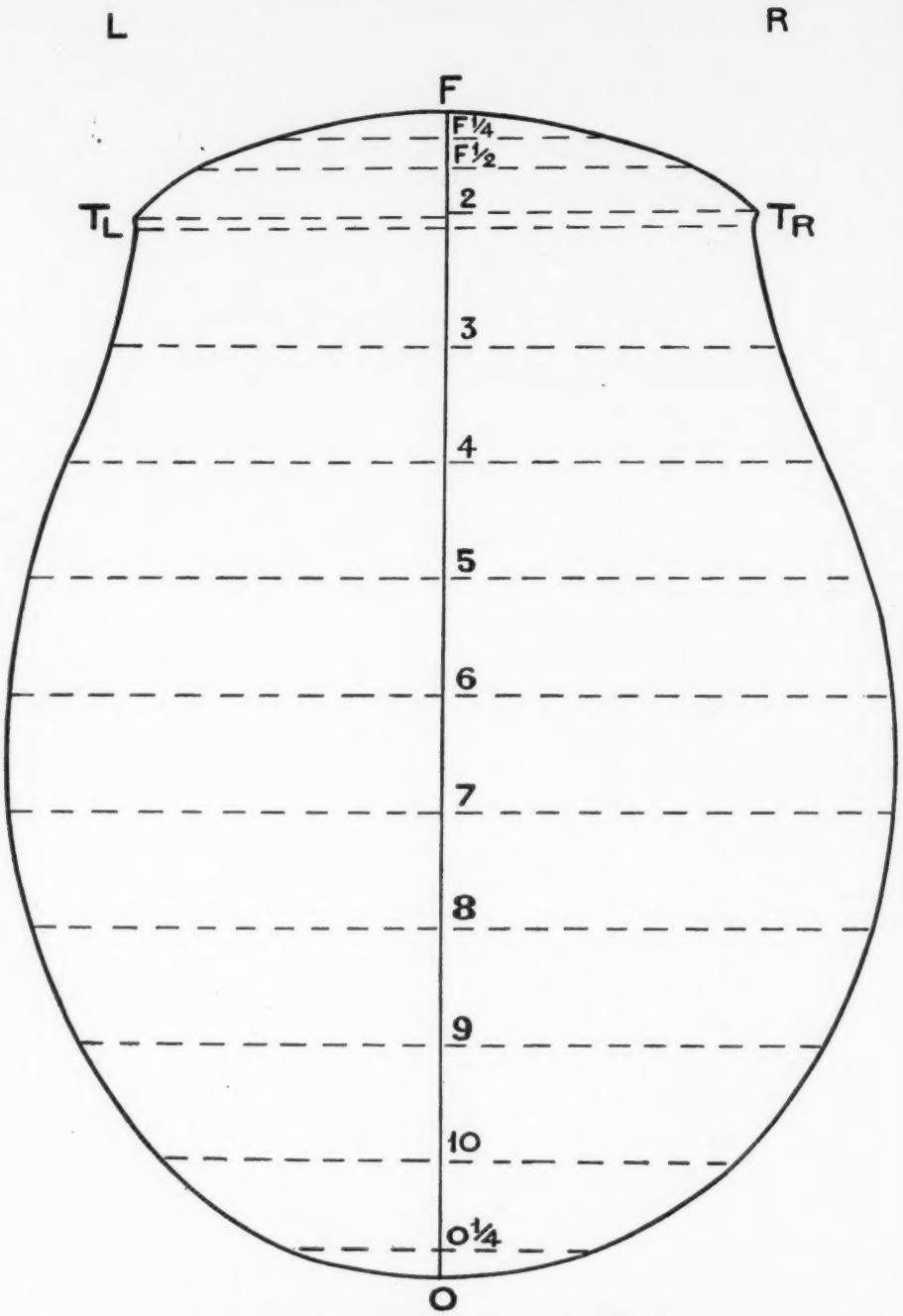


Fig. 4. Tibetan B. ♂. Horizontal Type Contour. (15 Crania.)

Considering first the two Tibetan types and making *FO* and the points *F* coincide, on the drawing of Type B and the tracing of type A show that they are of clearly differentiated forms. Type B is very significantly longer, actually by 10·1 mm., and the differences in the curvature of the frontal region are quite marked and particularly on the left side where the point $T(R)$ falls well outside the zone of Type A. In breadth the two are equal and if the tracing is moved down until the points *O* coincide the shapes of the occipital regions from *O* to parallel 6 are seen to be very similar. The Tibetan B horizontal section is characterised by very forward and prominent projections of the temporal lines which give an appearance of strength and flatness to the frontal bones. The lines $T(R)y$ and $T(L)y$ are well in front of the second parallel whereas they are usually behind it as with the Tibetan A, Moriori and Burmese types. The Tibetan A section also has the more common rounded form.

The Tibetan A contour is nearly equal in length to the three Burmese ones and very similar to them in all respects. The resemblance is particularly close to the Burmese A type which is nowhere more than 0·2 mm. from the superposed type zone. The correspondence with the Burmese series B and C is not quite so close, though it still suggests near relationship. The Tibetan B type approximates to the three Burmese only in breadth. It is much longer and the frontal regions are significantly wider. We can say without hesitation that the Tibetan A and Burmese A types are much more akin to each other than either is to the Tibetan B.

With a length 12 mm. less and a maximum breadth greater, the Tibetan A type can claim little affinity with the Moriori. The latter hardly falls within the type zone for a quarter of its length and perhaps it differs most characteristically in contracting behind the temporal lines where the Tibetan extends. The well-marked *linea temporalis* of the Moriori was taken to be a criterion of its primitive character*, and nothing so rugged can be found in any other type, but the position of these lines on the horizontal contour differs only slightly from that in which they are usually found and is not nearly so close to *F* as in the Tibetan B type. Is this prominence also a primitive character? It is unfortunate that the different method of construction obscures this region in the other racial contours. The Moriori skull is clearly differentiated from that of both the Tibetan peoples, but it resembles the B series more than the A.

The Tibetan A is in all cases significantly, and in many much shorter than the three Congo, the Egyptian, the Guanche and the English types; it is broader than all but the Guanche and English and it resembles none in shape except, perhaps, that of the third Congo series (Batetelu race) which falls entirely within the type zone, when the figures are superposed with *F* and *FO* coincident, with the exception of the occipital region below parallel 9. The Tibetan B contour is longer, broader and more massive than the three Congo and the Egyptian and shorter than the Eskimo, but it much more nearly resembles the Guanche and English and

* See *Biometrika*, Vol. xi. p. 104.

particularly the former‡. If the Tibetan and Guanche are superposed, with *FO* and either *F* or *O* coincident, the type zone will cover both contours for all but short lengths round the points $T(R)$ and $T(L)$. The difference in manner of construction may account for a small part of the discrepancies here, but, even if allowance is made for such, the characteristic prominence of the Tibetan type will still clearly distinguish it.

Our Tibetan A type then appears, from a comparison of horizontal contours, to be most closely related to the Burmese. There are two indices of interest which can be determined from the lengths of the horizontal type contours; they both give a measure of frontal development. The Temporal Index is formed by measuring the percentage Ordinate 3 is of the length of the section and the other index is Ordinate 3 divided by the minimum frontal breadth. The following table is taken from Thomson's paper with the addition of the Burmese and Tibetan data.

TABLE XV.

Measurements on the Horizontal Type Section. Males.

Race	Temporal Index	$100 \frac{\text{Ordinate 3}}{\text{Minimum Frontal Breadth}}$
Moriori ...	50·9	99·1
Burmese B ...	54·2	101·8
Burmese C ...	55·7	107·4
Egyptian ...	56·2	112·0
Tibetan B ...	57·0	107·0
Burmese A ...	57·1	102·3
Eskimo ...	57·2	[113·4 ?]
Congo, Bantu* ...	57·5	104·8
Guanche ...	58·3	[115·6 ?]
English ...	58·8	111·6
Tibetan A ...	59·1	107·7
Cro-Magnon ...	55·7	114·5†

* Fernand Vaz, 1864 Series.

† The Cro-Magnon minimum frontal breadth was determined from a cast.

The Burmese type has a distinctly low temporal index in spite of the fact that the skull is brachycephalic. The narrowness of the frontal bones behind the temporal lines is very characteristic of the race since it is found associated with a large parietal breadth and this is not a marked feature of the Tibetan A contour. But in spite of this point of difference we still maintain that the latter resembles the Burmese most closely and next the Congo Negroes. The B type resembles none of these, but differs least from the Guanche and then from the English.

Of the horizontal contours generally it may be noted that the divergencies between them are much greater than those observed when considering the vertical section and hence they will be more likely, when a sufficient number have been prepared,

‡ The similarity between the Guanche and the English was insisted on when the type contours were first compared: see *Biometrika*, Vol. VIII. p. 135.

to afford a reliable measure of racial affinity. But the sagittal section is undoubtedly more important than either of the others.

(*c*) *The Sagittal or Median Section.* With the nasion, bregma and lambda in the same horizontal plane, the sagittal contour is drawn through the alveolar point, nasion, glabella, bregma, vertex, lambda, gamma, inion, opisthion and basion and as much as possible is traced of the palate, nose and the basilar process of the occipital bone to the suture between it and the sphenoid; the vertex (i.e. the highest point in the median plane) and the gamma, or the point in the median plane on a horizontal level with the nasion, having been previously marked on the skull. All these points are dotted on the contour as well as the point where the palatine sutures cross*, the extremity of the *spina nasalis posterior*, the tip of the anterior nasal spine, the projection of the sub-orbital point and the point in the right auricular passage on which the skull must have rested when in the horizontal position on the craniophor—this is called the auricular point of the contour and the line joining it to the sub-orbital point should be parallel to that joining nasion to gamma (the $N\gamma$ line), if the two auricular points are symmetrically positioned. The two lines are very nearly parallel in the type contours. The horizontal $N\gamma$ line is taken as axis and divided into ten equal parts with the proportional compasses, the lines of division being numbered from 0, that through the nasion, to 10 through the gamma. Parallels 8 and 9 are produced below the base line to meet the drawing of the lower portion of the occipital bone and further ordinates are erected at $\frac{1}{4}$ the distance between the nasion and first parallel from the nasion ($N\frac{1}{4}$) and $\frac{1}{4}$ and $\frac{1}{8}$ of the same distance from the γ ($\gamma\frac{1}{4}$ and $\gamma\frac{1}{8}$ respectively). Also the ordinates 0, $N\frac{1}{4}$, I and II are produced below the base line to the palate. From all the points marked, with the exception of that marking the join of the palatine sutures, the basion and the alveolar point, ordinates (y) are dropped to $N\gamma$ and their positions are determined by taking either the nasion or gamma as origin of coordinates. As measures of curvature four maximum subtenses are taken: maximum frontal subtense to nasio-bregmatic chord, maximum occipital subtense to opisthio-lambdoid chord, maximum calvarial subtenses from the nasio-lambda and glabella-inion lines. The point of contact of a vertical tangent to the most projecting point of the occiput determines the occipital point.

The nasal bone of the contour is measured by joining the tip (L) to the nasion; NL giving the nose length and the angle between NL and $N\gamma$ being that of nasal prominence. Some nasal ridges exhibit a double curve, turning downwards towards the tip, so that for a short distance NL either cuts the outline or coincides with it. The point where NL first meets the outline of the ridge is called L' and NL' is also measured: where there is no double curve NL is, of course, equal to NL'. The maximum subtense is drawn from NL to the curve above L' and the point on the nasal bone is located from N with rectangular coordinates, x, along γN, and y.

The position of the alveolar point is determined by measuring its distances from the nasion and basion and the basion is measured from the gamma and the

* When the left and right transverse sutures did not meet in the median one, a point on that line mid-way between the two joins was taken.

nasion. To define more exactly the curvature between the anterior nasal spine and the alveolar point, a vertical tangent is drawn from the most posterior point of the arc. Finally the position of the point of the palate where the sutures cross is located from the alveolar point with rectangular coordinates and the subtense from the mid-point of the chord joining the basion and sphenoidal point is measured from the basion along that line.

These are all the measurements used in the construction of the type contour but a few others are taken with the object of comparing their mean values with the corresponding lengths and angles of the type diagram and also because some of them are wanted for comparative purposes. They are the lengths $N\beta, \lambda Op., N\lambda, GI$, the subtenses from $N\lambda$ and GI and the angles $\phi = \angle \beta GI$, $\phi' = \angle \beta NI$, $A\angle$, $N\angle$, $B\angle$, $P\angle$ and $fm\angle$ or the angle which the basio-opisthion line makes with the horizontal*.

Table XVI gives the mean values of the measurements made on the individual contours from which the types are constructed. The abbreviations used here and on the diagrams (Figs. 5 and 6) are: N = Nasion, G = Glabella, *Bas.* or B = Basion, β = Bregma, V = Vertex, A or *Alv.* = Alveolar point, *Sub-Orb.* = Infra-orbital point (left orbit), *Aur.* = Auricular point (right) on which the skull rested when in the Frankfurt horizontal position in the craniophor, λ = Lambda, γ = Gamma, I = Inion, *Op.* = Opisthion, *Sp.* = the Sphenoidal point, i.e. the point of intersection of the median plane and the suture between the basilar process of the occipital bone and the sphenoid bone, P = the point of intersection of the palatine sutures, P' = the extremity of the *spina nasalis posterior*, NS = the extremity of the anterior nasal spine, L = the tip of the nasal bone, L' = the point at which the line NL first meets the nasal bone, $A\angle = \angle NAB$, $N\angle = \angle ANB$, $B\angle = \angle NBA$, $P\angle = \angle$ between AN and γN, $\theta_1 = \angle$ between BN and $N\gamma$, $\theta_2 = \angle AB$ makes with the horizontal.

It should be noted here that when, as was occasionally the case, a point supposed to lie in the median plane (as the nasion, bregma, lambda, etc.) was either slightly below or above it, the pointer of the tracer was temporarily depressed or raised to pass through the point. It was seldom necessary to do this as most of the skulls were fairly symmetrical.

The comparative material for sagittal contours is the same as that for the other two type contours, but in this case many additional details have been added to the drawings since Benington's pioneer diagrams were made. The series we have numbered I to VII (p. 228) give the contour only from nasion to gamma and without indicating the position of any maximum subtense or of the vertex. This latter point was defined by Thomson in the Moriori paper, though not marked on her drawings, while she added the occipital arc from γ to opisthion, the inion, basion, alveolar, auricular and sub-orbital points, so giving the three vertices of the fundamental triangle. It then became possible to compare the angular measurements of the contours. Tildesley, for the Burmese crania, first drew the contour of the nasal bone and included the points P and V. We have indicated besides these the form of the palate and the arcs from alveolar point to anterior nasal spine and

* These measurements are given on p. 252.

from basion to sphenoidal point. It would hardly be possible to add further details without over-crowding and confusing the diagram.

In her Moriori paper* Miss Thomson has discussed at considerable length the relative worth of the many base lines which may be chosen as axis of reference when comparing sagittal type contours; alternatives fortunately not present in the case of the other two types. This point is of great importance if any clear meaning is to be given to such expressions as a "more receding forehead" and a "higher palate," or if comparisons between the positions of points such as vertex and bregma are to be made. A verdict was given in favour of the $N\gamma$ line which is used as axis in constructing the type contour. Its terminals are well defined and it represents very approximately the natural slope of the head. In accepting that as the common axis for most comparative purposes the use of other lines, such as $N\lambda$, $N\beta$, $NOp.$ or even $\beta\gamma$, need not be neglected, for the skull must not be considered to be dependent on one line. But no line terminating in the glabella or inion can be satisfactorily used because those points are not sufficiently well defined.

The sagittal type zone given by Benington is unfortunately only drawn from nasion to gamma; it has for the greater part of its length a width of 1·6 mm., tapering down to 0·1 mm. at the gamma and 0·9 mm. at the nasion. For 17 crania the corresponding widths will be 3·9 mm., 1·7 mm. and 2·2 mm., and for 15 crania 4·1 mm., 1·8 mm. and 2·3 mm. The type zones were drawn for the two Tibetan series, but owing to the danger of over-crowding are not shown on the type contours of Figs. 5 and 6.

The most marked characteristics of the Tibetan A median type section are the high forehead, flat glabella, and its smooth, almost feminine, curves. The B is a much more masculine type with a lower forehead than A and a quite distinct, though not prominent, glabella and inion. Superposing a tracing of A on the drawing of B with the $N\gamma$ line and the point N coinciding reveals at once the great difference in size between the two. In height they are very nearly equal, but the occiput of the B type protrudes massively beyond the other at all points and the whole palate is much lower, though very nearly parallel. From the nasion to as far back as the 8th parallel the Type B line, though it skirts the Type A zone for some way, is never more than 0·2 mm. away from it and this similarity in form of the frontal bones is made more apparent as far as the bregma if the tracing is moved round until the $N\beta$ lines correspond. But the occiputs still look of very different shapes.

In the three Burmese types the Tibetan A seems much more likely to find a counterpart of itself. Its differences from these are all small and in general appearance they are much alike. The resemblance is most close to the Burmese A and it must be remembered that the B and C are based on such small numbers of crania that a detailed comparison with other types is hardly warranted. Placing a tracing of the Tibetan A over the drawing of the Burmese A type, with the points N and the $N\gamma$ line coinciding, shows that the former is somewhat lower in the vault and more protruding in the occiput. But, keeping N fixed, if the tracing is now

* *Biometrika*, Vol. XI. pp. 105 *et seq.*

TABLE XVI.

Means of Contour Measurements used in plotting Sagittal Type Contours.

		Ordinates above $N\gamma$													
	$N\gamma$	$0=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	$\gamma=10$
Tibetan A (17)	171·7	27·5	41·6	60·3	72·5	79·5	84·4	86·6	86·4	82·7	73·6	55·4	27·3	19·2	5·9
Tibetan B (15)	181·7	23·6	38·3	59·1	71·8	79·2	83·6	86·1	85·5	80·4	70·4	51·5	23·6	15·7	−1·1*

	Ordinates below $N\gamma$								Vertex		β		Glabella	
	$0=N$	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y
Tibetan A (17) ...	68·4	65·4	55·9	53·3	46·4	35·7	28·5	19·9	90·5	87·2	70·6	84·6	2·5	12·9
Tibetan B (15) ...	73·1	68·6	59·7	57·4	51·5	41·7	35·6	26·9	96·4	86·5	75·4	84·0	3·0	12·0

	Occipital Point†		λ		Sub-Orb.		Aur.		Op.		Inion		Basion	
	x from γ	y	x from γ	y	x from N	y	x from γ	y	x from γ	y	x from γ	y	from γ	from N
Tibetan A (17)	−0·3	−3·6	6·0	30·6	6·9	27·9	87·9	27·7	51·3	51·0	9·0	29·2	98·9	95·8
Tibetan B (15)	−0·2	−8·1	6·4	28·0	6·4	29·7	95·7	29·4	58·3	57·1	8·6	35·8	107·6	99·3

	Alv.		Nose (16 Crania only for A series)					Palate (14 Crania for B Series)				Frontal Max. Sub. to $N\beta$		Occipital Max. Sub. to λ Op.	
						Max. Subt.		P'		P					
	from N	from B	NL	NL'	$\angle LN\gamma$	x from N	y	x from N	y	x from Alv.	y	x from N	y	x from λ	y
Tibetan A (17)	70·1	90·9	23·6	21·4	111·4	0·9	9·6	44·2	50·8	33·0	16·9	50·7	26·1	51·3	27·1
Tibetan B (15)	77·2	96·5	23·7	21·1	114·9	1·1	9·4	44·8	55·2	33·9	19·1	52·4	26·1	55·2	31·4

	Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Sub. from $\frac{1}{2}$ Bas. Sp. Chord		N. S.		Max. Sub. of Alv. N. S. Chord	
	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
Tibetan A (17)	87·1	70·5	96·0	98·7	63·2	30·7	12·7	1·9	3·4	47·0	2·2	58·9
Tibetan B (15)	87·8	71·8	97·2	101·2	64·1	33·2	13·5	1·8	4·9	53·6	4·1	56·9

* The negative sign here means that the ordinate 10 meets the contour below and not above the $N\gamma$ line.

† Both occipital points are below the $N\gamma$ line and to the left, with the type contours as shown, of γ.

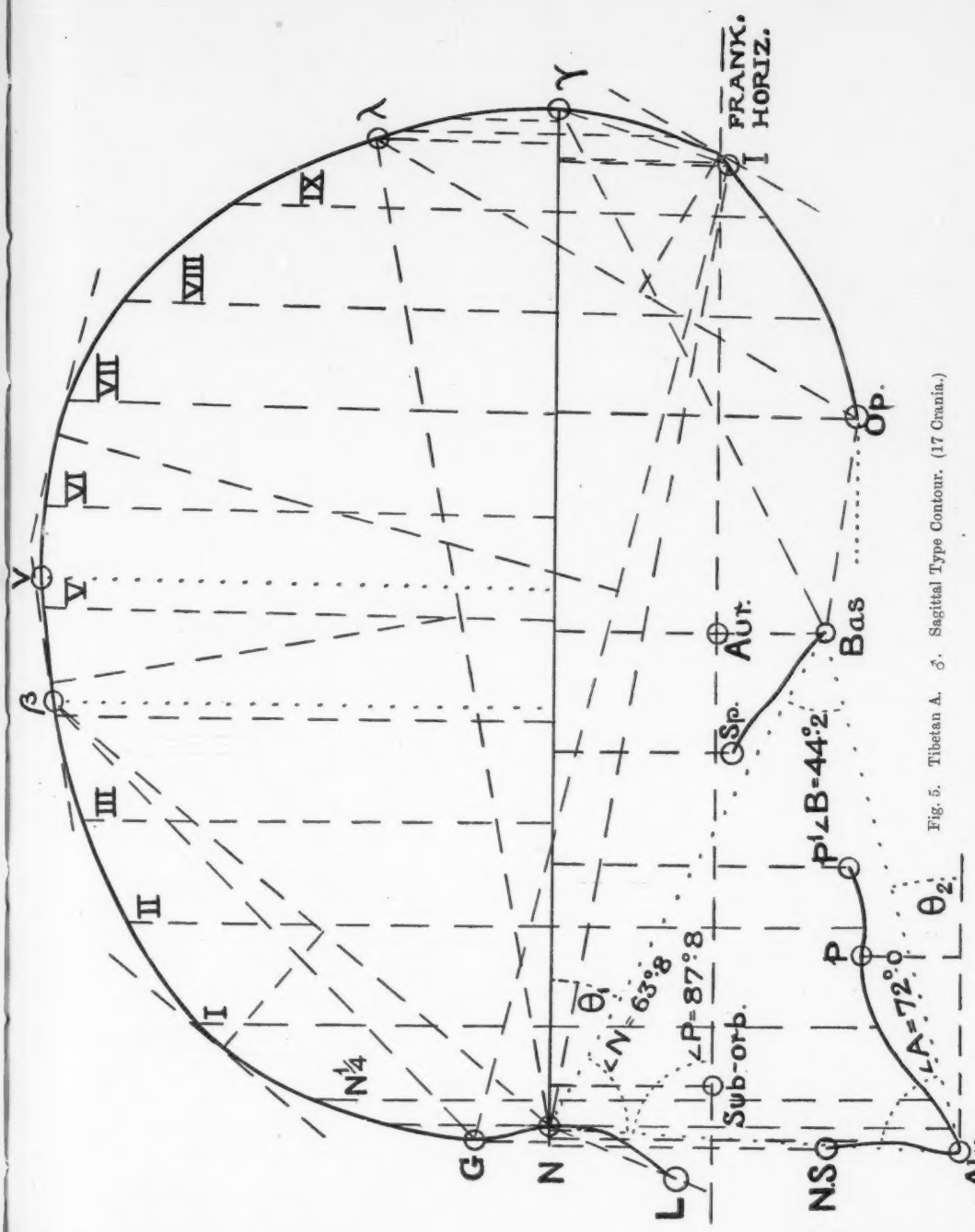


Fig. 5. Tibetan A. ♂. Sagittal Type Contour. (17 Crania.)

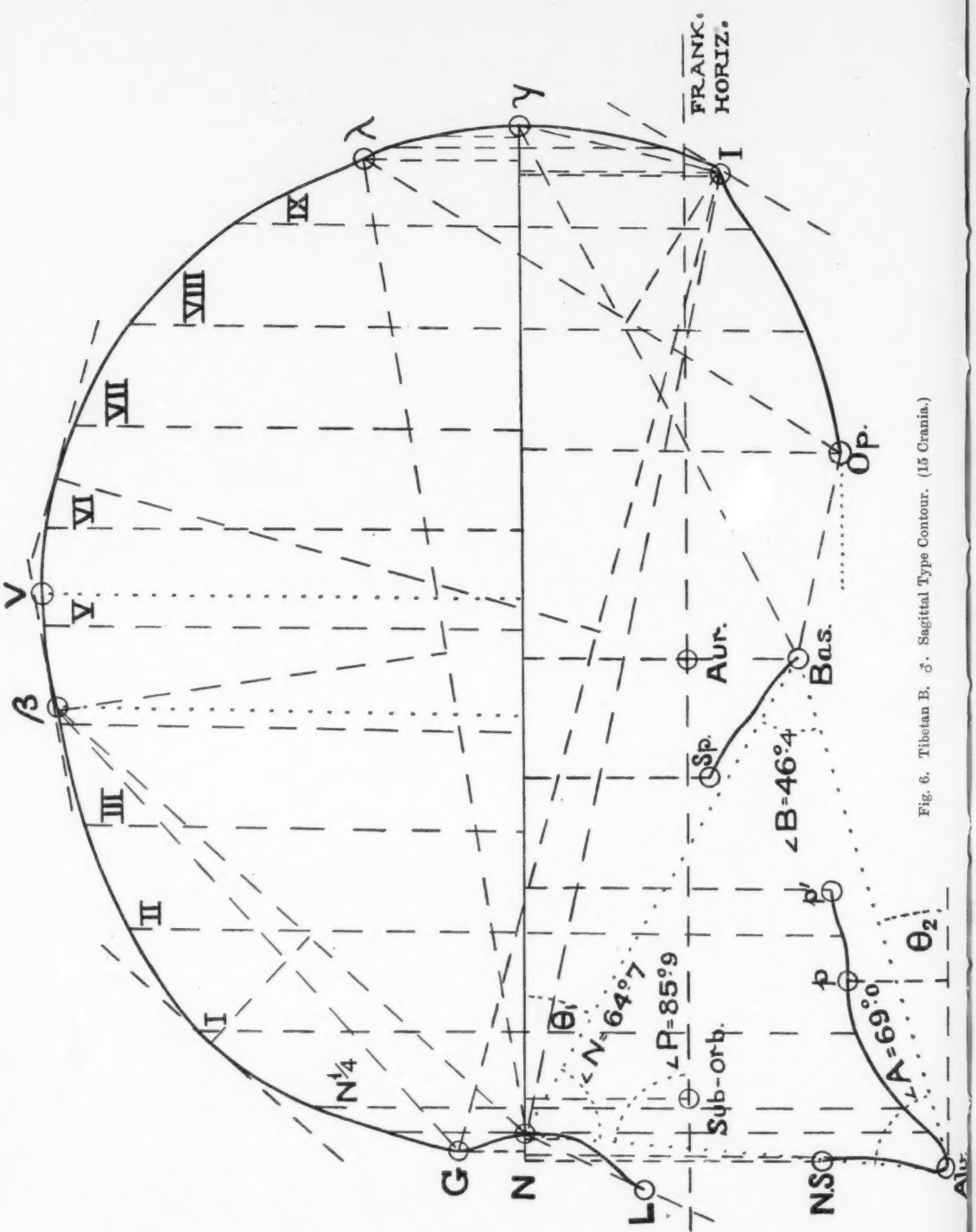


Fig. 6. Tibetan B. ♂. Sagittal Type Contour. (15 Crania.)

rotated to make the $N\beta$ lines coincide the resemblance becomes much more striking. From the tip of the nasal bone to the opisthion the type zone, though not shown below γ, must certainly cover both lines except a short distance between the inion and gamma of the tracing. The lines $N\lambda$ and $NOp.$ coincide exactly and the auricular, sub-orbital, and alveolar points, the basion and the point P are all remarkably close. The greatest divergencies are shown between the points gamma and the vertices. The correspondence is so striking and so much more marked in this position that we almost waver in our allegiance to the $N\gamma$ base line. With its lower vault, far greater length and distinctive occiput the Tibetan B is very unlike any of the Burmese types and, though almost equal in length, it has no other points of resemblance with "the depressed and retreating forehead, the bulging glabella and the high sagittal crest*" and the remarkably low γ of the Moriori section.

We have now to compare the Tibetan type contours with those drawn only from nasion to gamma. Our A section is shorter than the three Congo ones along the $N\gamma$ line by 3·9, 6·2 and 4·7 mm. respectively, but almost exactly of the same height and with a frontal bone very similar to theirs. This latter point can be most clearly seen if the $N\beta$ rather than the $N\gamma$ lines are superposed. The correspondence is then very exact, and well within the limits of the type zone, from the nasion to the 6th parallel with the first Congo series, to the 5th parallel with the second series, and the 5th again with the third. That the resemblance is not more general can be seen by making the $N\gamma$ lines and the points γ coincide; the less convex Tibetan occiput clearly differentiates it from the others. There is little to suggest kinship between the Tibetan A and the remaining races. The former is much shorter and at the same time higher than the Egyptian, Guanche and English and not unlike them and the Eskimo posteriorly, but it is most like in the anterior region. This correspondence is again surprisingly close when the $N\beta$ lines of the contours are superposed and much less so when the $N\gamma$ line is the common base. That such should have been the case with all the types compared, with the single exception of the Moriori, is a very noteworthy fact.

The Tibetan A is greater in $N\gamma$ length than any of the three Congo types; in height much the same but differing from them in the higher and more bulging form of its glabella. With $N\gamma$ as axis of reference and the points N coincident, the frontal bones are again seen to have almost identical curves and this correspondence with the first Congo section extends well behind the bregma so that the type zone covers both contours from the nasion to a point not far from the lambda. The discrepancies between each of the three Congo types and the Tibetan B are greater with $N\beta$ than with $N\gamma$ as base line, the latter being unlike its compatriot A type in that respect. If either of these lines is chosen as axis of reference, or perhaps one intermediate between them, the Tibetan B contour is found to have a frontal bone which is hardly distinguishable in shape or size from those of the Egyptian, Eskimo, Guanche and English and this was found with all the other types, with the exception of the Moriori. A parallel statement was made when comparing the

Fig. 6. Tibetan B. ♂. Sagittal Type Contour. (15 Crania.)

* Thomson, *Biometrika*, Vol. XI. p. 89.

Tibetan A type with these races. The occipital region, on the other hand, is often distinctive and hence more likely to be of use in differentiating racial types. The glabella too is often characteristic, being decidedly higher above the nasion on the Tibetan B type than on those of the Guanche, English and Eskimo. In other respects these correspond very closely and particularly in the form of occiput and vault. The Egyptian, though still very similar, has a frontal bone more unlike that of the Tibetan than any of the others, except the Moriori. The English and Guanche are most like this Asiatic race in the absolute size of the section, but is that to be our sole criterion? Superposing the Tibetan B tracing on the Eskimo with $N\gamma$ as base line does not at first reveal any similarities except in the form and position of the glabella; the contour tracing is entirely surrounded by the drawing. By rotating into the $N\beta$ position, the angles between that line and $N\lambda$ are seen to be identical and one frontal bone is, as far as the bregma, almost entirely covered by the other. Can we now place the one as symmetrically as possible inside the other so as to obtain some appreciation of their similarity in form apart from mere size? This can be effected by placing the nasion of the Tibetan A type 2·2 mm. from that of the Eskimo (the difference in $N\gamma$ length is 4·4 mm.) and keeping that point fixed while the tracing is moved round until the nasion of the tracing and the two lambdas are collinear. The contours are then seen to be very approximately parallel from the N to the γ of the circumscribing one; the greatest distance between them is 2·5 mm. and the least 1·5 mm. Does this denote racial affinity? Not even a tentative answer could be given to such a question without more abundant comparative material; contours for which the regions below the $N\gamma$ line are mapped out being particularly required. The comparison made with the Eskimo type shows, at any rate, how easily a real similarity in shape of head may be overlooked, if attention is to be confined to positions fixed by common base lines and it makes us question again the invariable use of the $N\gamma$ horizontal line for that purpose. Sagittal type contours appear to be potentially of far greater importance than the other two, but at present, owing to the uncertainty attached to the methods of comparing them, they provide a less reliable racial criterion than the horizontal ones.

In Thomson's Moriori paper (pp. 105—112) there is a detailed discussion of the relative values of the nasio-gamma, nasio-lambda and glabella-lambda base lines when each is judged according to its capability of giving an estimate of the flattening of the frontal bone and of physiognomic flatness; a verdict was given in favour of the nasio-gamma line. Accepting that base line, we can now add the Burmese and Tibetan data to the comparative table (*loc. cit.* p. 108) of the subtenses of the calvaria.

The interesting features of Table XVII are seen in columns 7 and 9. The very rounded form of the Burmese and Tibetan A crania, as shown by the high y indices, clearly differentiates them from all the other types, including the Tibetan B. But there seems to be nothing racially distinctive in the positions of the vertex and bregma, as measured by the x coordinates, or in the differences between the indices given in columns 11 and 12. The bregmatic angles are rather more characteristic,

TABLE XVII.

Measurements on the Sagittal Type Section. Males.

Race	$N\gamma$	Max. Subtense		β Subtense		Index Max. Subtense		Index β Subtense		Δ (Ind. Max. Sub. – Ind. β Sub.)		Bregmatic Angle
	B_0	y	x	y	x	y/B_0	x/B_0	y/B_0	x/B_0	y/B_0	x/B_0	
English ...	183·5	84·9	103·7	83·0	77·2	46·3	56·5	45·2	42·1	1·1	14·4	47°·1
Guanche ...	183·4	84·8	99·0	83·0	74·4	46·2	54·0	45·3	40·6	0·9	13·4	48°·1
Egyptian ...	181·9	84·4	106·2	82·5	79·0	46·4	59·5	45·4	43·4	1·0	16·1	46°·2
Tibetan B ...	181·7	86·5	96·4	84·0	75·4	47·6	53·1	46·2	41·5	1·4	11·6	48°·2
Moriori ...	180·2	90·0	103·8	84·9	73·0	49·9	57·6	47·1	40·5	2·8	17·1	49°·3
Congo, Bantu*	175·6	85·4	97·7	83·5	70·6	48·6	55·6	47·6	40·2	1·0	15·4	49°·8
Eskimo ...	186·1	93·9	104·1	89·3	72·1	50·5	55·9	48·0	38·7	2·5	17·2	51°·1
Tibetan A ...	171·7	87·2	90·5	84·6	70·6	50·8	52·7	49·3	41·1	1·5	11·6	50°·1
Burmese C ...	172·3	88·0	98·4	86·0	73·3	51·1	57·1	49·9	42·5	1·2	14·6	49°·6
Burmese B ...	170·9	90·5	100·5	86·5	70·1	53·0	58·8	50·6	41·0	2·4	17·8	50°·7
Burmese A ...	167·4	90·7	91·8	88·2	69·5	54·2	54·9	52·7	41·5	1·5	13·4	51°·8
Cro-Magnon ...	196	100·5	103·2	97·7	68·2	51·3	52·7	49·8	34·8	1·5	17·9	55°·2

but they fail to distinguish between the Oriental races and the Eskimo and Bantu.

Thomson has suggested† as a measure of physiognomic flatness the bregmatic angle for the nasio-gamma line combined with the angle the line joining the nasion to the arc end of the nasio-bregmatic subtense makes with the nasio-bregmatic line. The contributions of each of these to frontal flattening are shown in the following table.

Physiognomic Angle of Flatness.

Race	Frontal Bone Flatness	Rotation of Base of Frontal Bone	Physiognomic Angle of Flatness
Egyptian ...	24°·9	46°·2	71°·1
Moriori ...	22°·3	49°·3	71°·6
English ...	26°·6	47°·1	73°·7
Guanche ...	26°·4	48°·1	74°·5
Tibetan B ...	26°·5	48°·2	74°·7
Burmese C ...	25°·2	49°·6	74°·8
Burmese B ...	24°·2	50°·7	74°·9
Congo, Bantu*	27°·3	49°·8	77°·1
Tibetan A ...	27°·3	50°·1	77°·4
Burmese A ...	26°·7	51°·8	78°·5
Eskimo ...	28°·0	51°·1	79°·3
Cro-Magnon ...	24°·8	55°·2	80°·0

The Physiognomic Angle of Flatness does not appear to distinguish the Tibetan A and Burmese types from the remainder as well as the bregmatic angle does.

* Fernand Vaz, 1864 Series.

† *Biometrika*, Vol. XI. p. 100.

Having made comparisons between the corresponding type contours of various races it is natural to ask how similarities noticed when considering one section are correlated with those adduced from another. And, further, if some judgment of racial affinity can be obtained from all three sections, how will this be correlated with an index based on direct measurements? A tentative answer can be given to the first of these questions. If comparison is confined to members of the same family of races—such a family as the one to which the Oriental races are supposed to belong—then it appears to be probable that all three sections of the two types being compared will be similar if one section is found to be so. But, though one contour of two unallied races may display a striking similarity, it by no means follows that the other two sections will correspond at all closely. The Tibetan A transverse type contour is almost identical with those of both the Moriori and Whitechapel English, but the horizontal and sagittal types are as dissimilar as any.

A comparison of the racial contours has led us to conclude that the Tibetan A and the three Burmese types, particularly the B, are closely related, but the former does not resemble the Tibetans of Kham more than it does the Congo Negro or Whitechapel English type. The Tibetan B appears to be unlike all the Oriental races with which we are able to compare it, but its calvaria is very similar in size and shape to the English calvaria.

We may now ask how the Coefficients of Racial Likeness are correlated with the judgments of racial affinity based on comparisons of the three type contours. Unfortunately we are only able to make this comparison in the case of a few races; these are the two Tibetan races, the three Burmese, the Moriori, Congo Negroes (Batetela) and Whitechapel English.

Of the 31 direct, indicial and angular measurements from which the Coefficients of Racial Likeness are calculated there are only 18 which are either actually represented, or very nearly represented, on the type contours. These are *B*, *Q*, *OH* and *B*/*H*, corresponding to the transverse vertical section; *B'*, *U* and *B*/*L'* to the horizontal and *F*, *LB*, *S*, *G'H*, *NB*, G_1, *fml*, *Oc. I.*, $P\angle$, $N\angle$ and $A\angle$ to the sagittal section when all the points we have indicated are marked on that.

There seems at first to be very little correlation between the judgments of racial affinity arrived at by comparing type contours and the corresponding Coefficients of Racial Likeness. The Tibetan A and Burmese B direct measurements appear to be very similar and the coefficient between them is low, but there was no striking resemblance between the type contours. On the other hand, the contour and coefficient methods of comparison both suggest that there is a closer relationship between the Tibetan A and Burmese than there is between the Tibetan A and the Tibetan B or any other of the Oriental races. It was surprising to find that all the Tibetan B and Whitechapel English sections were very similar; the Coefficients of Racial Likeness, however, suggest no affinity. The English sagittal type contour was given only from the nasion to the gamma, so that there was no representation at all of the facial regions and base of the skulls, and of the 31 characters from which the coefficient was calculated there are only 7 lengths,

angles or indices which could be measured on the three contours. The C. L. for these 7 characters alone is 2·96, but when all 31 are considered it becomes 11·93. The great differences between the English and Tibetan B skulls are between the facial and palatine measurements; only the calvariae of the two are similar.

A contour of a skull gives, of course, a more detailed description of the plane in which it is taken and the zones in the immediate neighbourhood of that plane than any number of direct measurements can do, but the three contours cannot give a complete picture; they give no representation at all of the orbits and only a very incomplete one—when all possible points have been marked—of the remainder of the face. The Coefficient of Racial Likeness takes into account all regions of the skull. It would thus seem that a comparison of type contours, though a study of great interest, can never furnish a very reliable measure of racial affinity and it is particularly liable to lead to erroneous deductions when unallied races are compared.

9. *Comparison between Direct Measurements and Contour Values.*

The causes of the discrepancies found between contour and direct measurements have been discussed in detail by Tildesley in her Burmese paper*. We will now compare our results by the two methods.

The defect in the Klaatsch contour tracer, noted by Miss Tildesley and to which she partly attributed the discrepancies found between her measurements, had been remedied before I used the instrument; also the drawing-board on which the

TABLE XVIII.

Comparison of Direct and Contour Mean Measurements†.

		Tibetan A	Tibetan B
Glabellar-occipital Length (*L*) ...	Direct	175·2 (17)	185·7 (14)
	Sagittal Contours	174·8 (17)	185·2 (14)
Upper Face Height (*G'H*)	Direct	69·4 (17)	76·5 (15)
	Sagittal Contours	70·1 (17)	76·5 (15)
Nasion-Basion Length (*LB*) ...	Direct	95·7 (17)	99·2 (15)
	Sagittal Contours	95·8 (17)	99·3 (15)
Basio-Bregmatic Height (*H'*) ...	Direct	131·2 (17)	134·1 (15)
	Sagittal Contours	131·1 (17)	133·9 (15)
Length of Foramen (*fml*)	Direct	35·7 (17)	37·4 (15)
	Sagittal Contours	36·9 (17)	38·4 (15)
Basion to Alveolar Point (*GL*) ...	Direct	91·5 (17)	97·2 (15)
	Sagittal Contours	90·7 (17)	96·4 (15)
Opisthion to Lambda ($\langle S_3' \rangle$)	Direct	94·4 (17)	100·3 (15)
	Sagittal Contours	94·5 (17)	100·0 (15)

* *Biometrika*, Vol. XIII. Nos. 2 and 3, pp. 210 *et seq.*

† The figures in brackets are of the number of skulls on which the means are based.

tracing was made appeared to be quite level and not in any way warped, at any rate no flaw could be detected by simple tests. After a considerable time had been spent in practising and checking the accuracy of the drawings by comparing several made of the same skull, a fairly reliable method of manipulation was attained.

In some few cases Miss Tildesley deduced the direct and contour means from different numbers of crania, one pair differing by 2 and the others by 1. All the series dealt with were small and it was thought best to exclude no measurement that could possibly be made. Fortunately our Tibetans were in a much better state of preservation than the Burmese and all corresponding means have been based on the same numbers.

The contour means in the following table are those reduced from measurements made on the individual tracings and they may not be precisely equal to the lengths of the type contours.

The greatest differences between the means found by the two methods are for the length of the *foramen magnum*, but the lengths from basion to alveolar point and the upper face heights are not in very close agreement. The majority of the discrepancies are so small that we cannot attribute the larger ones to instrumental defects or constant errors in manipulation, for such would affect all the measurements in much the same way. A sufficient explanation seems to be given if we remember the well-known difficulties of determining cranial "points." These were not marked on the skulls when the direct measurements were taken and they may easily have been located differently on the contours. The well-known vagueness of the alveolar point would account for the rather large discrepancies observed for the measurements $G'H$ and GL, since they are both taken from it. The marked discrepancy in the case of the length of the *foramen magnum* is probably due to the fact that the direct measurement was taken further inside the orifice than the contour. On the whole we can feel justified in placing considerable reliance on the accuracy of the type contours.

Why should the two series of Tibetan means be more consistent than those of the Burmese? The greater differences found for the latter must have been chiefly due to the defective tracer and unlevel drawing-board used. Another possible source of error lies in the marked asymmetry of many of the Burmese skulls. In drawing the sagittal contour it was often found necessary to raise or depress the pointer of the tracer in order to make it pass through a point which had to be marked. It was seldom necessary to do this with the Tibetan skulls as there was little asymmetry among them.

Several angular measurements found in the two ways are compared in Table XIX.

The greatest difference here is 0°·8 so that the agreement is sufficiently close, for practical purposes.

The auricular point was marked on the sagittal contours and a line through it perpendicular to the $N\gamma$ horizontal will give a measure of the height, OH, of the individual contours, if the two auricular points of the skull were symmetrically

TABLE XIX.

Comparison between Mean Angles from Direct Measurements and Mean Angles from Individual Contours.

		P ∠	*N* ∠	*A* ∠	*B* ∠
Tibetan A	Direct Measurements	87°·4 (15)	64°·7 (16)	71°·8 (16)	43°·5 (16)
	Sagittal Contours ...	87°·7 (15)	63°·9 (16)	72°·1 (16)	44°·1 (16)
Tibetan B	Direct Measurements	85°·7 (14)	65°·5 (15)	68°·8 (15)	45°·7 (15)
	Sagittal Contours ...	85°·6 (14)	64°·7 (15)	69°·0 (15)	46°·4 (15)

placed, but we should not expect it to agree very closely with the direct measurement. The same length can be found from the individual transverse vertical contours* by dropping a perpendicular from the marked vertex on to the line joining the points which are supposed to have been those in contact with the ear-rods of the craniophor. But here again we should expect no close correspondence.

TABLE XX.

Tibetan Mean Auricular Height, OH.

	Direct Measurements	From Sagittal Contours	From Transverse Contours
Tibetan A (17)	113·2	114·2	113·7
Tibetan B (15)	115·5	115·1	116·0

The conclusion to be drawn from the above comparisons is that direct and contour mean measurements are in reasonably close accordance and should differ by less than 1 mm.

Several measurements are taken from the individual sagittal contours which are not used in the construction of the type and a comparison of the means of these with the corresponding lengths and angles of the type contours is of interest. The following tables give our results.

The greatest difference between these pairs of measurements is 0·4 mm. Several angular measurements can be compared in the same way and the differences will again be found very small. We are dealing with a short series of skulls and the discrepancies would certainly tend to be smaller for a longer series, so we can say confidently that the method of construction of sagittal type contours if special care be used is sufficiently accurate for comparative purposes.

* As much as possible of the auricular passage was traced, though not all could be represented on the type contour.

TABLE XXI.

Comparison between measurements on Type Contours and means of Individual Contour measurements which are not used in Construction of Type Contours.

		Tibetan A (17)	Tibetan B (15)
$N\beta$	Mean Contour Value	110·2	113·2
	Type Contour	110·2	113·5
λ Op.	Mean Contour Value ...	94·4	100·0
	Type Contour	94·0	100·2
$N\lambda$	Mean Contour Value	168·7	177·7
	Type Contour	169·0	177·8
β subt. to $N\lambda$	Mean Contour Value	70·2	71·3
	Type Contour ...	70·4	71·0
GI	Mean Contour Value	170·5	182·3
	Type Contour	170·3	182·7
β subt. to GI	Mean Contour Value	87·5	90·2
	Type Contour ...	87·4	89·4

TABLE XXII.

Comparison between Angle-measurements on Type Contours and means of Individual Contour measurements.

		Tibetan A (17)	Tibetan B (15)
ϕ	Mean Contour Value ...	58°·8	58°·1
	Type Contour ...	59°·0	57°·6
ϕ'	Mean Contour Value ...	60°·5	59°·9
	Type Contour ...	60°·7	59°·5
$A\angle$	Mean Contour Value ...	72°·0	69°·0
	Type Contour ...	72°·2	69°·3
$N\angle$	Mean Contour Value ...	63°·8	64°·7
	Type Contour ...	63°·8	64°·5
$B\angle$	Mean Contour Value ...	44°·2	46°·4
	Type Contour ...	44°·0	46°·2
$P\angle$	Mean Contour Value ...	87°·8	85°·9
	Type Contour ...	87°·6	85°·8
$fm\angle$	Mean Contour Value	8°·8	12°·7
	Type Contour ...	8°·3	12°·3

10. *Conclusions.*

It seems reasonable to conclude that:

(i) There are in Tibet at least two distinct races.

(ii) One of these (viz. that from the southern provinces in the neighbourhood of Sikkim—our Type A) is closely allied to the Southern Chinese, Malayans and Burmese.

(iii) The other race (the Khams Tibetans), though showing no marked affinity with any Oriental race, resembles most the Burmese B and C types. The skull is, however, very similar to those of both Fuegians and Moriori, and the impression cannot be avoided that we may possibly be dealing with widely scattered fragments of a fundamental primitive human type, with a long-headed, broad-faced, rugous and massive cranium.

I should like to acknowledge here the help of Miss Ida McLearn in drawing the contour diagrams and of Mr E. S. Pearson in photographing the crania. I have also to thank Dr W. L. H. Duckworth for several suggestions and corrections to this paper.

APPENDIX I. *On the Measurement of the Mandible.*

These measurements on the Tibetan mandible are placed here with a view to more elaborate future treatment of mandibular characters of various races and especially to provide adequate data for comparisons with the mandible of palaeolithic man.

Of the 32 Tibetan crania which have been dealt with in the first section of this paper 16 of the A series and 7 of the B were provided with mandibles. In addition to these the collection included 5 mandibles of the B type which had not been allocated to any of the remaining skulls and it could not be said definitely that they corresponded with any of them. Thus, altogether, there were 16 mandibles of the A type and 12 of the B which could be measured. When dealing with the mandibles of any particular race the craniometrician has often to be content with numbers that are not larger than this, while the mandibles are often separated from the skulls and cannot be sexed with any certainty; no reliable determination of type can be adduced from such material. Consequently the measurements of the mandible have been neglected by most workers and when any are given they are invariably scanty.

The following measurements have been taken for the Tibetan series. Many of the lengths and their methods of measurement have been considered by Broca and Rudolf Martin, but some are new.

(*a*) *Definitions of Mandible "Points."*

The Frankfurt horizontal plane is, for obvious reasons, not one to which the measurements of the detached mandible can be conveniently referred. When the mandible is resting freely, with the teeth uppermost, on a horizontal surface its position will usually be unique, but occasionally it may be capable of a slight rocking displacement. Accordingly, the plane of reference to which the majority of heights and angles we shall speak of will be related is defined to be that horizontal plane with which contact is made at three or more points when a vertical pressure is applied to the second left molar tooth or its cavity: this will be known

as the standard basal plane, and when pressed vertically downwards on the second left molar the mandible will be said to be in the "standard basal position*." Usually the three points or regions of contact will be near to the right and left angles and a point on the left corpus somewhere below the canine or premolar teeth, but such is not invariably the case. One or even both angles may not be touching. The basal plane is certainly of a more conventional nature than the Frankfurt horizontal plane, but we are obliged to adopt it because, as far as we can see, no other plane could make the technique of measurement so exact or be less ambiguously defined. An attempt will be made to measure on the individual crania the angles at which the standard basal plane of the mandible and the Frankfurt horizontal plane are inclined ($S \angle$).

Certain mandibular "points" of osteometric rather than anatomical importance have to be defined. The majority of these have been discussed by earlier writers†, but we have found it impossible to accept certain of their definitions without modification. We doubt, indeed, whether they have been tested in practice on any adequately long series.

It must always be understood, unless otherwise stated, that the mandible is in the "standard basal position."

Coronion (c_r). Two coronia are required, one on either ramus. In a perfectly symmetrical jaw one plane could be placed in contact with both condyles and both coronoidal processes, but with the normal asymmetry this is impossible. Accordingly the left coronion is defined by the point of contact of a plane touching the left condyle, left coronoidal process and either right condyle or right coronoidal process as the case may be. The right coronion is obtained by proper interchange. These points, as well as the two condylia, are determined by inverting the mandible on to a flat sheet of duplicating paper so that the points of contact are marked. If there are two summits of equal height the apex of the more forward is taken. When a measurement is taken to only one of the coronia it is always that of the left ramus unless otherwise stated.

Condylion (c_y). This is defined and found in practice in a way similar to that described in the case of the coronion, the capitulum and coronoidal process being interchanged. If a small area of the capitulum instead of a single point is in contact with the above tangential plane then the "centre" or midpoint of that area is termed the condylion and it can be found with sufficient accuracy from the impress left by the duplicating paper.

Gnathion (g_n). The lowest point in the median or symphysial plane when the mandible rests in the standard basal position. This is invariably on, or very close to, the common tangent to the anterior borders of the *fossae digastricae*.

* Unilateral measurements are accordingly taken on the left side, but if, owing to breakage, a measurement has to be taken on the right ramus or corpus, then the second right molar is held to give the standard position.

† See especially, Rudolf Martin, *Lehrbuch der Anthropologie* (1914) (Mandibular "points" are defined, pp. 516—518, and measurements and methods of measurement, pp. 559, 560, 565, 566), and Broca, *Instructions Craniologiques et Craniométriques* (1875), pp. 93—96.

Pogonion A (p_a). The most projecting point of the chin, or, as it is termed by English anatomists, the mental prominence, when the mandible rests in the standard basal position.

Pogonion B (p_b). The most projecting point of the chin when the mandible is *in situ* on the skull adjusted to the Frankfurt horizontal plane.

Infradental (d_f). The point where the sagittal plane of the mandible meets the horizontal tangent to the alveolar rims of the middle incisors.

Intradental (d_t). The tip of the process between the middle incisors.

Gonion (g_o). This is, perhaps, one of the most important points of the mandible and, at the same time, it is the most difficult to define and to determine. A definition of the standard basal plane has been given above. It is now necessary to define the standard rameal plane which we do as follows: Let a tangent plane be taken to the left ramus touching it at the condyle and towards the angle, and let it also touch the right ramus where it can (if the mandible were symmetrical it would touch at the right condyle and at a point towards the right mandibular angle). This tangent plane is the standard rameal plane as defined from the left ramus. It may also be defined from the right ramus. The intersection of the standard basal and rameal planes is the gonional axis. The nearest points on the borders of the angles to the gonional axis are the gonia. The gonia are found in practice with the aid of the mandible board described below. The mandible is held on the second left molar and adjusted so that the rameal wing of the board is in contact with two points on the left ramus and one on the right. Then the zero line of the board corresponds with the gonional axis and the points on the two angles nearest to it are the right and left gonia respectively*.

* The gonion was first defined and named by Broca (*op. cit.* p. 93). In practice the point can be found easily and without ambiguity and it appears to fulfil the ideal condition, that it should be the point of intersection of the posterior line of the ramus and the inferior line of the corpus, as well as any point could. Martin (*op. cit.* p. 517) has proposed quite a different method to determine it. The mandible is inverted so that the angles are uppermost and so that the left ramus and corpus make equal angles with the horizontal. Then the gonion is the highest point of the angle. I marked this point on several mandibles and found that in all cases it was in front of the gonion determined by Broca's method and that for several the distance between was as much as 8 mm. The personal equation of Martin's method must certainly be large as neither of the operations by which the point is determined can be performed with any precision.

We are unable, also, to accept Martin's definitions of some other "points." The pogonion and gnathion, for example, are defined without indicating any plane of reference. They are marked on two figures (*op. cit.* pp. 505, 517) showing the *norma lateralis* and *norma sagittalis* of the complete skull with the mandible in its natural position. It appears from this that the pogonion has reference to the Frankfurt horizontal plane and not to one on which the free bones would rest, so it corresponds to our pogonion B. Martin suggests taking many measurements from this point and defines no other in its neighbourhood. In practice this is found to be a very unsuitable terminal for many reasons: it is impossible to locate the point if the cranium is either missing or badly damaged as is frequently the case; if the skull be available but some, or all, of the teeth are missing it may not be possible to place the mandible in the natural position, though some approximation to that can be made by supporting the mandible with plasticine; damaged or decayed condyles will introduce a difficulty of the same kind and, finally, even if the bone be complete, it is often impossible to place the mandible in any unique position which can be supposed its natural one. For these reasons the pogonion B, though in itself a point of great importance, appears to us to be a very unsuitable one to take measurements from and

The positions of all the points defined above are marked on the bone in pencil before any measurements are taken from them.

Another mandibular plane has to be defined. The standard sagittal plane of the mandible is the plane perpendicular to the standard basal plane, passing through the intradental and perpendicular to the bigonial axis; it should also contain the infradental point, the two pogonia and the gnathion.

(*b*) *The Mandible Board.*

All the measurements, with the exception of those made with small callipers, Flower's callipers, or the steel tape, are taken when the bone is resting, in one position or another, on the mandible board; an instrument similar in design to Broca's "goniomètre mandibulaire*." A fixed horizontal board (*A*) has hinged to it at one end a movable board (*B*)—to be called the "rameal wing" of the instrument—so that the latter can be inclined at any angle from 0° to 180° with the horizontal. The angle between *A* and *B* can be read off directly from a semicircular scale attached to a vertical board (*C*) so that the centre of the scale is in a line with the centre of the hinge. Both the rectangular boards *A* and *B* have mm. scales along one edge fixed so that the centre of the hinge is their common zero. With the aid of this instrument several mandibular angles can be found directly. The bone rests on the horizontal board *A* and is held on the second left molar and heights of points above that plane, which is that of the standard basal plane, are determined by adjusting the point of a scriber to their level and then reading off the height of the latter above the basal plane on the scale attached to the board *B*, when that is in a vertical position. A solid set-square is provided for the purpose of measuring the projections of lengths. This consists of two lengths of wood, of which the section is square, fastened together at one of their extremities, but not let into each other, so that they are at right angles. If one of these branches is resting on the mandible board parallel to the hinge, the other will be parallel to the length of the board, i.e. the edge to which the scale is fastened and with its upper surface on a line with it. Then the solid set-square is slid along until it comes into contact with the point on the mandible from which the measurement has to be made; the other extremity of the length is the zero line, i.e. the central line of the hinge. The projected length can be read off from the horizontal scale. The heights of the coronoidal and condylar processes can be found in a similar way by using the board with the rameal wing vertical.

(*c*) *Measurements and Methods of Measurement.*

The following is a list of the direct measurements of the mandible which I have determined for the Tibetan series. All measurements, unless otherwise stated,

accordingly, the more conventionally defined pogonion A—the point always referred to below unless the other is specifically named—has been used for that purpose. It makes only a slight difference to the position of the gnathion whether it is defined with reference to the Frankfurt horizontal or standard basal plane, but the two pogonia may be several mms. apart.

* See *Instructions Craniologiques*, p. 95.

are given for the left side only. The lengths w_1, w_2*, h_1, and zz were provided by Fawcett for the Naqada Egyptians and have been determined for several other races by subsequent workers in the Biometric Laboratory.

(i) *Lengths measured with callipers or tape.*

w_1, condylar width. Greatest width of mandible at condyles from outside of one condyle to outside of second; with small callipers. w_2, greatest width of mandible at angles from outside of one angle to outside of the other; with small callipers. h_1, sagittal height of mandible; measured from gnathion to intradental†. zz, least distance between inner rims of *foramina mentalia*. $c_r c_r$, coronial breadth from coronion to coronion. rb, least breadth of ramus parallel to basis. As a guide to measurement a pencil line parallel to the basis is drawn on the ramus in the region of its least breadth while it is orientated on the standard basal plane. rb', least breadth of ramus in any direction. G_2', breadth of mandible between inner alveolar walls at middle of second molars. This is the counterpart of the measurement G_2 of palate breadth. $c_y c_r$, condylion to coronion. $g_o g_o$, gonion to gonion. $g_n g_o (l)$, gnathion to left gonion. $g_n g_o (r)$, gnathion to right gonion. $c_y l$, greatest length of condyle. $c_y b$, greatest breadth of condyle perpendicular to $c_y l$. $m_2 p_1$, distance between outer alveolar margin at the middle of the second molar to middle of first premolar. $p_a d_t$, pogonion to intradental. $p_a g_n$, pogonion to gnathion. $p_a d_f$, pogonion to infradental. $g_n d_f$, gnathion to infradental. $p_a p_b$, pogonion A to pogonion B. $g_o p_a g_o$, bigonial arc, measured with the steel tape from one gonion to the other through the pogonion; a very uncertain measurement.

(ii) *Heights and Lengths; using Mandible Board.*

Unless otherwise stated, it is always to be understood that the mandible is in the standard basal position. Heights above the plane of the basis are found either by the aid of a scrib-awl or with the solid set-square, and projections are found with the solid set-square (see p. 256).

ih, least height of incisura; with scrib-awl. ih', greatest depth of incisura from a line joining condylion to coronion. The mandible is inverted so that it is supported on the left side by the condylion and coronion and by one of those points on the right ramus. Then the measurement is the greatest height of the incisura above the supporting surface; read by aid of scrib-awl. $c_r h$, height of coronion. Projection on to vertical rameal wing of board, using solid set-square. $c_y h$, condylar height. Vertical projection as for $c_r h$. $d_t h$, height of intradental; with scrib-awl. $m_2 h$, height of outer alveolar margin at middle of second molar; with scrib-awl. $p_1 h$, height of outer alveolar margin at middle of first premolar; with scrib-awl. $c_p l$, length of corpus. The rameal wing of the mandible board is brought up into the position in which it touches two points of the left ramus and one of the right. The measurement $c_p l$ is the projection from the zero line of the board to the most

* See *Biometrika*, Vol. I. p. 417. Fawcett gave the letters W_1 and W_2 to these two measurements, but Thomson altered them to w_1 and w_2 as W seemed more appropriate for weight of skull. I have used small letters to denote all mandibular measurements.

† This and all the following lengths are measured with small callipers unless otherwise stated.

advanced point of the chin—the pogonion A. *rl*, length of ramus. Projection of ramus on to rameal wing of board with position as for finding $c_p l$. *ml*, mandibular length. Projection from posterior points of condyles to pogonion A, with rameal wing of board vertical.

(iii) *Angular Measurements using Mandible Board and Craniophor.*

$M\angle$, mandibular angle, i.e. angle between the standard basal and rameal planes; position on board as for finding the length $c_p l$. $R\angle$, angle of condylar-coronoidal line with ramus tangent. The mandible is inverted so that it rests on the left coronion and condylion and one point of the right ramus. Then adjustment is made so that the rameal wing of the board is in the position of the left ramus tangent while making contact with one point on the right ramus. $G\angle$, gnathio-gonional angle, i.e. the angle subtended at the gnathion by the two gonia; determined from the measured lengths $g_o g_o$, $g_n g_o(l)$, and $g_n g_o(r)$ with the aid of Pearson's Trigonometer. $C\angle$, angle between the standard basal plane and the line in the sagittal plane joining pogonion A to intradental; found with the goniometer resting on a horizontal plane parallel to the standard basal plane. $C'\angle$, angle between standard basal plane and the line joining pogonion A to infradental; similar to $C\angle$. $C\angle$ and $C'\angle$ are the "mental angles" of the mandible. $L\angle$, angle between line joining pogonion A to infradental and Frankfurt horizontal plane; measured with goniometer on skull and mandible adjusted on craniophor to Frankfurt horizontal with base of skull uppermost. The angle is measured as on the left *norma lateralis* (the skull being vertex uppermost) from the facial profile clockwise to the horizontal, so $L\angle$ will be obtuse if the infradental is more advanced than the pogonion. $L'\angle$, angle between line joining pogonion B to infradental and Frankfurt horizontal plane; measured with goniometer in same sense as $L\angle$. $L\angle$ and $L'\angle$ are the "mandibular profile angles." $F\angle$, angle between line joining pogonion B to nasion and Frankfurt horizontal plane. This is the total profile angle (*angulus lateralis totalis*). $S\angle$, the angle between the Frankfurt horizontal and standard basal planes; this is $(L\angle - C'\angle)$ and no great reliance can be placed on its accuracy since $L\angle$ is rather an uncertain measurement.

(iv) *Indicial measurements.*

From the foregoing direct measurements the following indicial ones are calculated: $100\, c_r h/ml$, height-length; $100\, c_r c_r/ml$, breadth-length; $100\, g_o g_o/c_p l$, corpus breadth-length; $100\, rb'/rl$, rameal breadth-length; $100\, c_y b/c_y l$, condylar breadth-length; $100\, g_o g_o/c_r c_r$, gonional-coronoidal breadths; $100\, c_y h/c_r h$, condylar-coronoidal heights; $100\, ih'/c_r c_y$, incisural depth-length; $100\, d_t h/c_r h$, intradental-coronoidal heights.

The mean measurements of the Tibetan mandibles are given in the following table.

As we have so far no comparative material it is not possible to estimate the true significance of the differences between the mean mandibular measurements of the two Tibetan series. The B type, which has the larger and more muscular and rugous skull, exceeds the A type for the majority of mandibular lengths, it is equal to it

TABLE XXIII. *Tibetan Mandibular Mean Measurements* (mms.).

TABLE XXIII. *Tibetan Mandibular Mean Measurements* (mms.).*

		Lengths with Callipers														
		w_1	w_2	h_1	zz	$c_r c_r$	rb	rb'	G_2'	$c_y c_r$	$g_o g_o$	$g_n g_o(l)$	$g_n g_o(r)$	$c_y l$	$c_y b$	$m_2 p_1$	$p_a d_t$
Tibetan A ...	Mean	117·8	95·3	30·7	45·1	94·6	36·8	31·4	47·6	34·8	92·9	83·4	83·7	18·6	7·9	29·3	25·9
	No.	16	16	15	16	16	16	16	16	16	16	16	16	16	16	15	15
Tibetan B ...	Mean	122·0	99·4	35·9	47·0	93·9	37·3	34·2	48·6	38·1	96·5	88·7	88·0	19·9	8·2	29·2	31·2
	No.	12	12	12	12	12	12	12	12	12	11	11	12	12	12	12	12

		Lengths with Callipers				Tape	Lengths and Heights on Mandible Board									Indices	
		$p_a g_n$	$p_a d_f$	$g_n d_f$	$p_a p_b$	$g_o p_a g_o$	ih	ih'	$c_r h$	$c_y h$	$d_t h$	$m_2 h$	$p_1 h$	$c_p l$	rl	ml	$c_r h/ml$	$c_r c_r/ml$
Tibetan A ...	Mean	7·7	23·0	28·2	3·6	187·4	40·4	16·3	60·7	49·6	34·8	23·3	28·4	74·0	57·8	105·1	57·8	89·8
	No.	16	15	15	16	16	16	16	16	16	15	16	16	·16	16	16	16	16
Tibetan B ...	Mean	7·8	28·1	33·8	4·0	201·0	43·6	16·6	65·2	54·9	40·1	27·6	33·9	78·9	61·0	109·6	59·7	86·0
	No.	12	12	12	7	12	12	12	12	12	12	12	12	12	12	12	12	12

		Indices							Angles							
		$g_o g_o/c_p l$	rb'/rl	$c_y b/c_y l$	$g_o g_o/c_r c_r$	$c_y h/c_r h$	$ih'/c_r c_y$	$d_t h/c_r h$	$M\angle$	$R\angle$	$G\angle$	$C\angle$	$C'\angle$	$L\angle$	$L'\angle$	$F\angle$	$S\angle$
Tibetan A ...	Mean	125·8	54·5	42·7	98·4	81·7	46·9	57·6	126°·1	71°·4	67°·7	64°·4	61°·4	96°·8	95°·1	91°·7	35°·4
	No.	16	16	16	16	16	16	15	16	16	16	15	15	15	15	16	15
Tibetan B ...	Mean	121·7	56·3	41·3	102·9	84·0	43·8	61·7	123°·0	73°·1	66°·2	66°·7	63°·1	101°·1	99°·8	95°·5	38°·0
	No.	11	12	12	11	12	12	12	12	12	11	12	12	7	7	7	7

* The values given in this table for the lengths $g_o g_o$, $g_n g_o(r)$ and $g_o p_a g_o$ are not theoretically correct, though they are close approximations to the true values, because they were not determined by the position of the right gonion in the way described above (p. 255). The right gonion should be the point on the right angle nearest to the zero axis of the mandible board when the mandible is held on the second *left* molar and the rameal wing of the board is in contact with two points on the *left* ramus and one on the right ramus. Instead of this point I marked, and took measurement from, the point on the right angle which was nearest to the zero axis of the board when the mandible was held on the second *right* molar and the rameal wing was in contact with two points on the *right* ramus and one on the left ramus. This point must be very close to the true gonion.

for a few measurements and markedly less for none. Some of the mean characters of the two types differ by more than 5 mm. and these differences appear to be greater, in proportion, for heights than for horizontal lengths. The indices are very similar and perhaps their differences have no significance, but the angles appear to be more characteristic; these show that the B type has the more receding chin when the Frankfurt horizontal is the plane of reference ($L\angle$, $L'\angle$ and $F\angle$) and consequently a less prominent one than the A type when resting on the standard basal plane ($C\angle$ and $C'\angle$).

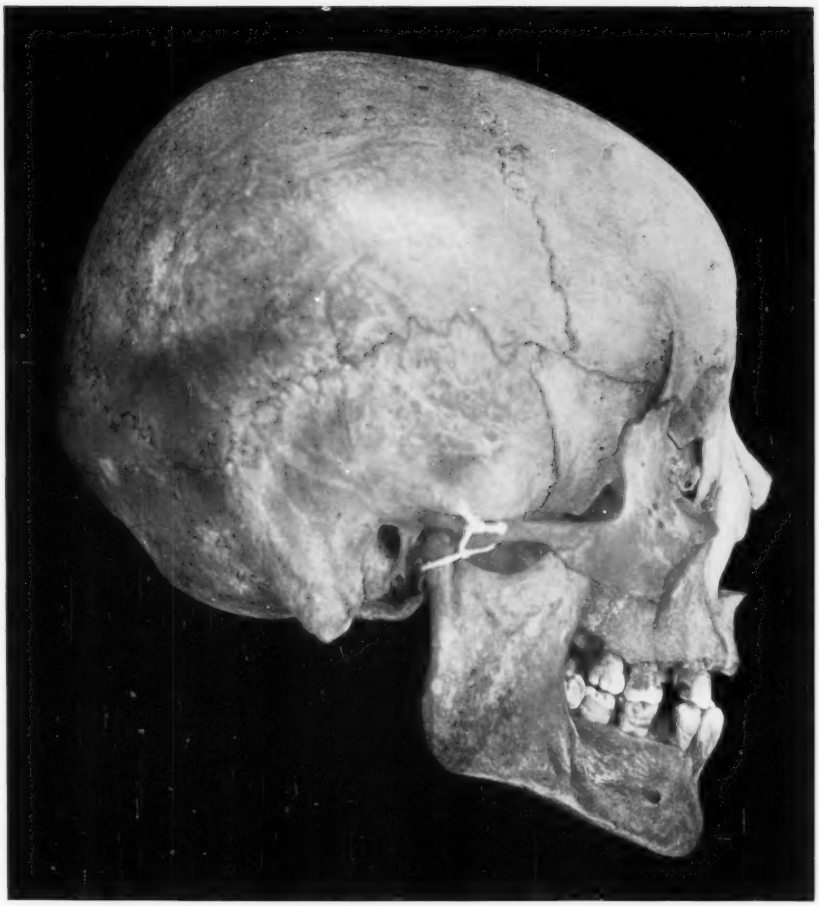
DESCRIPTION OF PLATES.

Plates I to X give the five aspects of two typical skulls of the Tibetan A and B types respectively. Neither the *norma lateralis* nor the *norma facialis* brings out clearly the greater facial flattening of the skulls of the A type which was one characteristic which distinguished them from the skulls of the B type.

Plate	I,	Tibetan	A, No. 29, *Norma lateralis.*
,,	II,	,,	B, No. 27, ,, ,,
,,	III,	,,	A, No. 29, *Norma facialis.*
,,	IV,	,,	B, No. 27, ,, ,,
,,	V,	,,	A, No. 29, *Norma basalis.*
,,	VI,	,,	B, No. 27, ,, ,,
,,	VII,	,,	A, No. 29, *Norma verticalis.*
,,	VIII,	,,	B, No. 27, ,, ,,
,,	IX,	,,	A, No. 29, *Norma occipitalis.*
,,	X,	,,	B, No. 27, ,, ,,

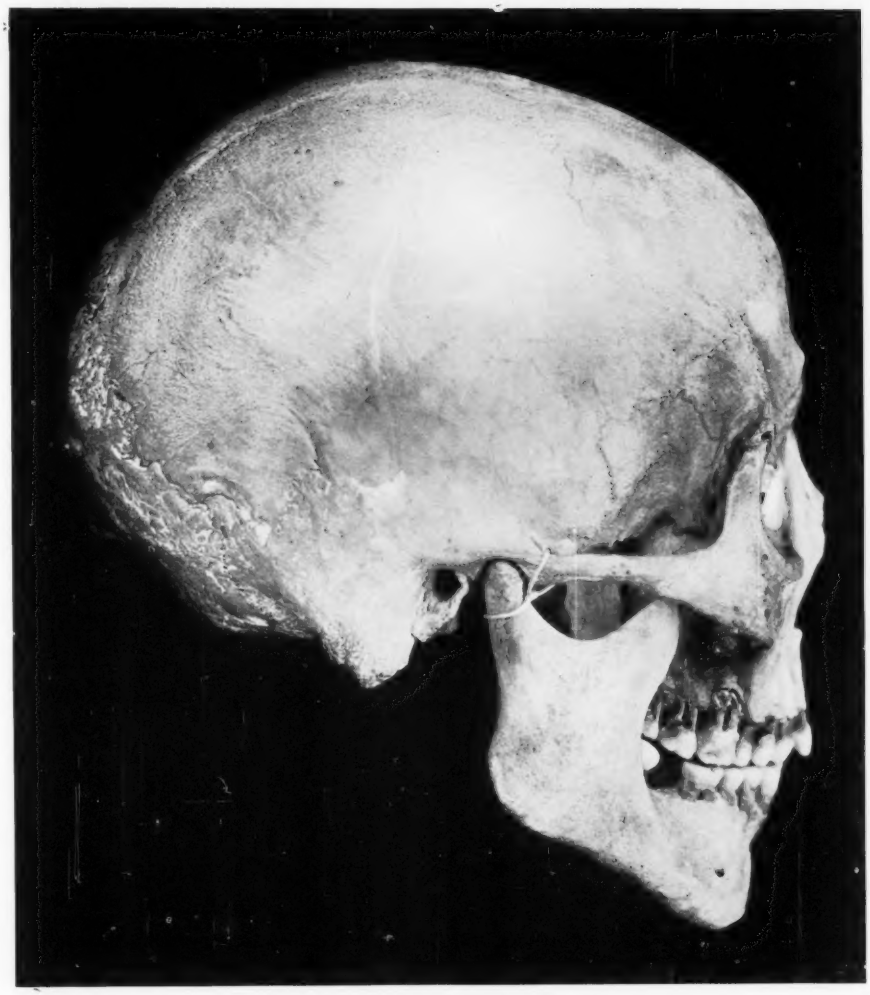
The remaining plates (XIV (right) excepted) illustrate anomolies.

Plate	XI	(left),	Tibetan	B, No. 30, *Os triangulare.*
,,	,,	(right),	,,	A, No. 25, Bipartite *os triangulare.*
,,	XII	(left),	,,	B, No. 19, Bipartite interparietal.
,,	,,	(right),	,,	A, No. 1, Ossicles of lambdoid suture.
,,	XIII	(left),	,,	A, No. 8, Asymmetrical precondyles.
,,	,,	(right),	,,	B, No. 17, Single precondyle.
,,	XIV	(left),	,,	A, No. 3, Asymmetrical precondyles.
,,	,,	(right),	,,	B, No. 14, This photograph shows how common tangent to *lineae nuchae superiores* passes nearly through superior inion.



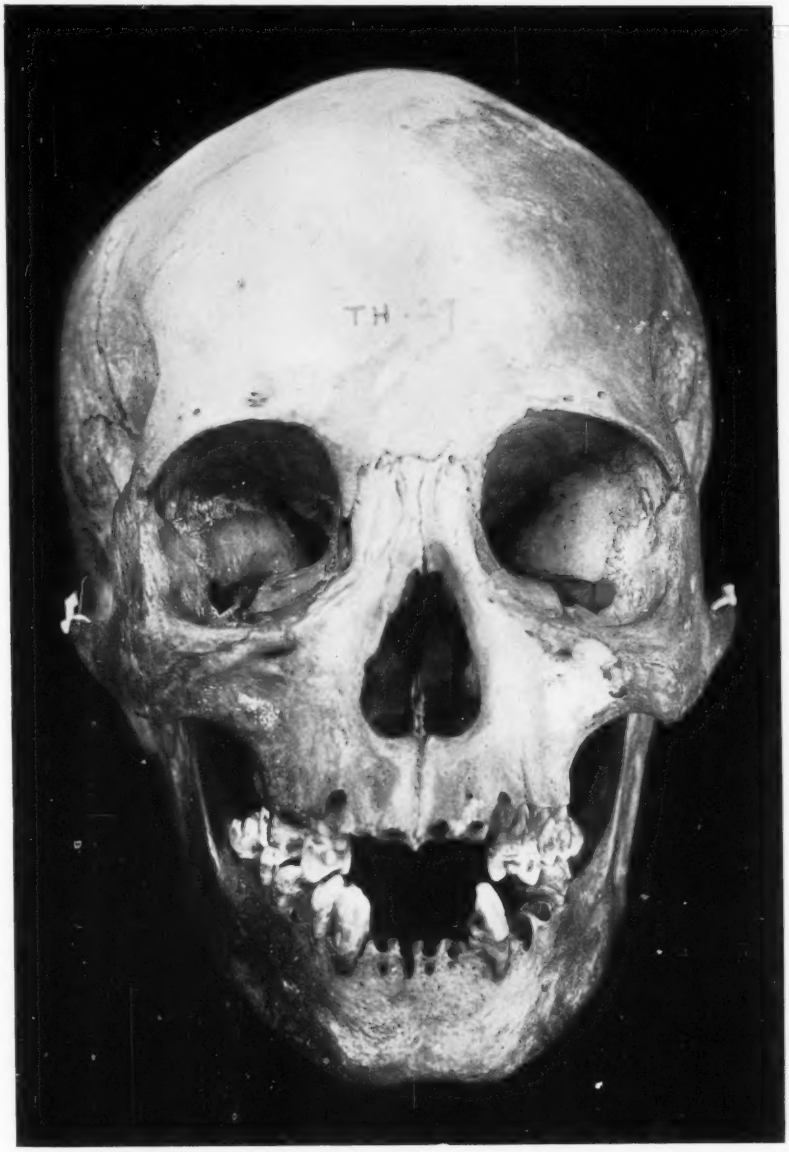
Tibetan Skull, Type A, No. 29.

Norma lateralis.



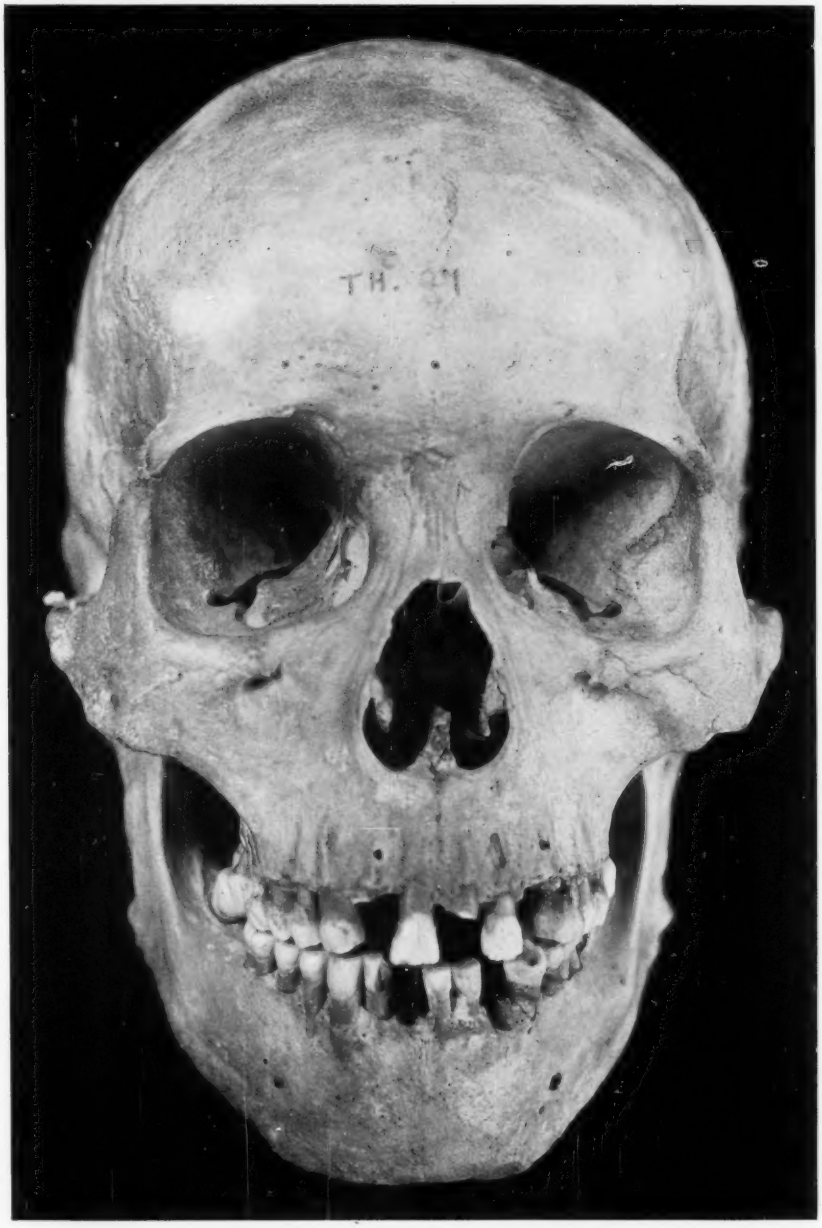
Tibetan Skull, Type B, No. 27.

Norma lateralis.



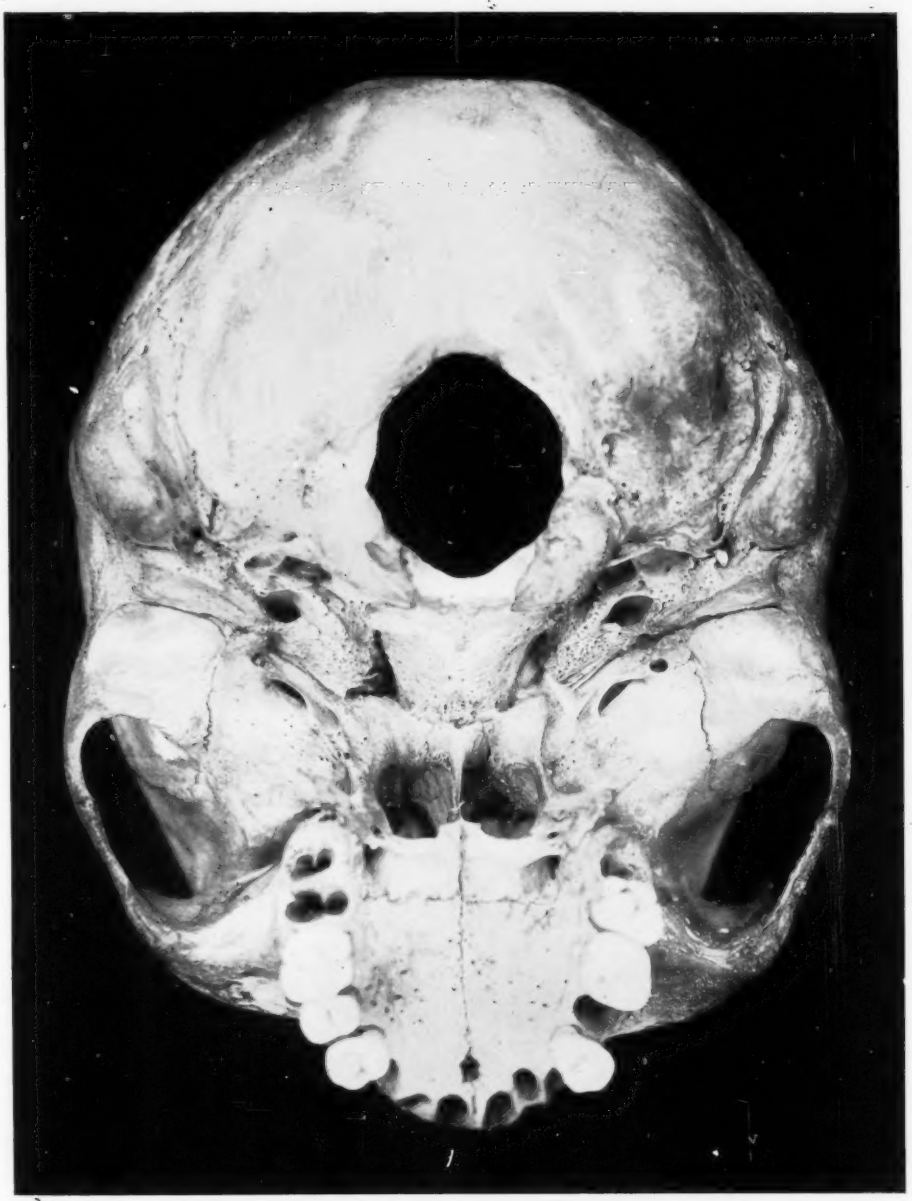
Tibetan Skull, Type A, No. 29.

Norma facialis.



Tibetan Skull, Type B, No. 27.

Norma facialis.



Tibetan Skull, Type A, No. 29.

Norma basalis.

Tibetan Skull, Type B, No. 27.

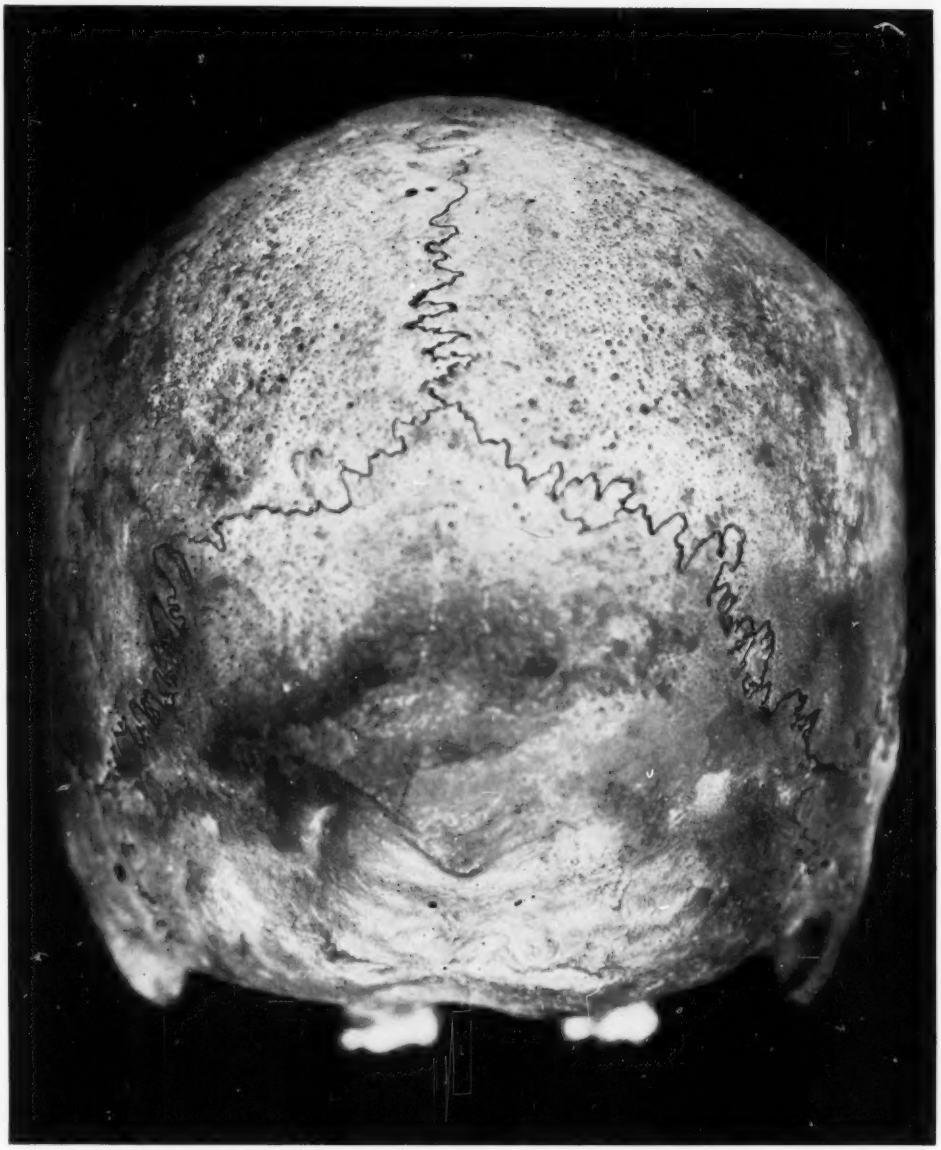
Norma basalis.

Tibetan Skull, Type A, No. 29.

Norma verticalis.

Tibetan Skull, Type B, No. 27.

Norma verticalis.



Tibetan Skull, Type A, No. 29.

Norma occipitalis.

Tibetan Skull, Type B, No. 27.

Norma occipitalis.

Special Skull, No. 30.

An os triangulare replaces the more usual os pentagonale.

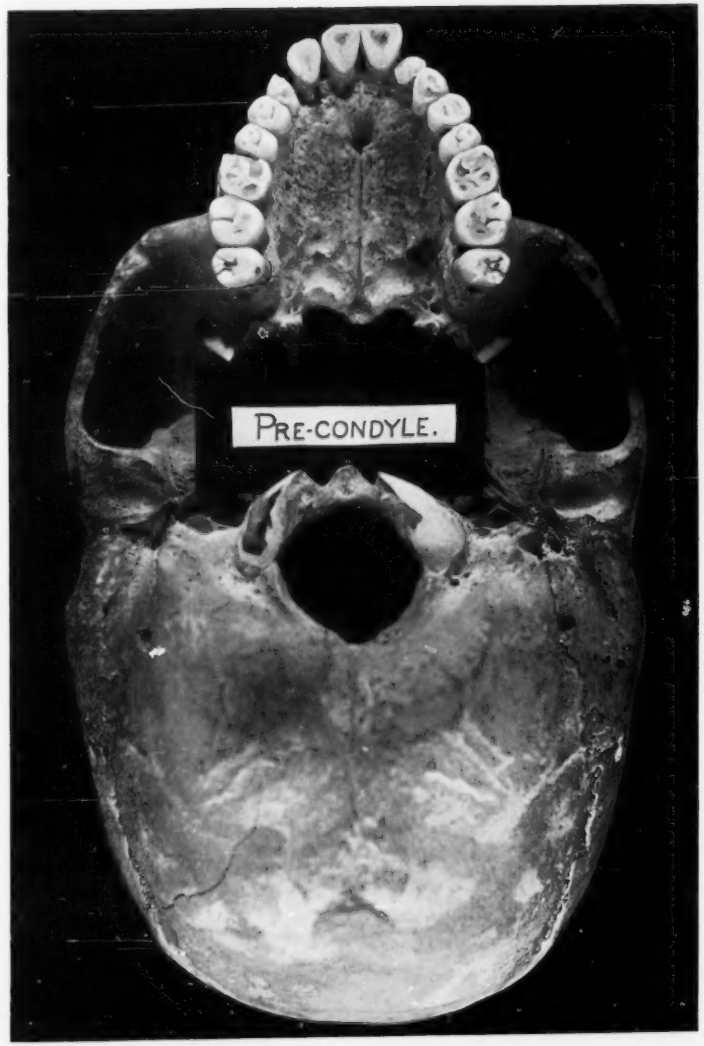
Special Skull, No. 25.

The os triangulare replacing the os pentagonale becomes bipartite.

Special Skull, No. 1.
Numerous Ossicles of lambdoid Suture.

Special Skull, No. 19.
Bipartite Interparietal.

Special Skull, No. 8.



Special Skull, No. 17.

Special Skull, No. 3.

Asymmetrical Precondyles.

Special Skull, No. 14.

Showing how common tangent to lineae nuchae superiores passes nearly through superior inion.

					Lengths											
No.	Type	Sex	*C*	W. in grs.	*F*	*L′*	*L*	*B*	*B′*	*H*	*H′*	*OH*	*LB*	*Q*	*Q′*
1	A	♂	1846·5	661·6	188·5	191·5	190·5	152·5	94·5	145	144	121	102	332	333
2	,,	♂ (?)	1390	323·5	173	173·5	172·5	132	96·5	125·5	126·5	113	94	301	300
3	,,	♂ (?)	1215·5	389·6	166·5	166	167	130	86	125·5	127	106·5	90·5	288	289
5	,,	♂	1554	387·0	170	170	169·5	150·5	95	133·5	130·5	118	92	329	33[illegible]
6	,,	♂	1534	435·4	176·5	175·5	175	138·5	99·5	126	125·5	114	93	315	316
7	,,	♂	1563·5	603·5	178·5	175·5	177·5	142·5	92	142·5	142	118	96·5	321	323
8	,,	♂	1459·5	588·2	179	179·5	179·5	142·5	94·5	127·5	127	110·5	94·5	304	304
9	,,	♂	1393	649·4	178·5	180	179·5	138	93·5	133·5	131·5	112·5	93	307	309
10	,,	♂	1471·5	390·0	173·5	173·5	173·5	135·5	98	130·5	129	108·5	96·5	301	303
16	,,	♂	1318·5	496·6	160	163·5	162·5	139	92·5	131	130	113·5	94·5	315	314
20	,,	♂	1372·5	619·2	176	179	178·5	135	95	136	135	114·5	99·5	314	315
21	,,	♂	1421·5	545·0	168·5	172	171·5	148·5	94·5	132·5	132	108·5	100	313	309
22	,,	♂	1437·5	686·2	178·5	181·5	180·5	139	92·5	132	130·5	113	102·5	305	308
25	,,	♂	1488	712·2	183	183·5	183	149	90	121·5	121	109	91	308	314
28	,,	♂	1111	577·8	169	170·5	169·5	130	81	131	130	112·5	93·5	304	303
29	,,	♂	1457	475·6	174·5	172	172	140	91·5	136	135	116	95	312	313
32	,,	♂ (?)	1327	390·2	171	171	170	126·5	87·5	134·5	133·5	115·5	99	309	300
4	B	♂	1454	756·6	180·5	182·5	182·5	136	97·5	138·5	139	116·5	98	311	310
11	,,	♂	1406	601·0	180	179·5	179·5	135	94	128·5	127·5	112	96	302	301
12	,,	♂	1493·5	941·2	191·5	192	191	140·5	94	140·5	139	123·5	96·5	329	332
13	,,	♂	1651	620·4	182	185·5	184·5	141·5	95·5	134·5	133·5	114·5	95	315	313
14	,,	♂	1482	801·4	179·5	182·5	181	142	92	132·5	130·5	113·5	96	310	310
15	,,	♂	1370	421·4	171·5	173·5	173·5	136·5	91	126	127	108	91·5	291	293
17	,,	♂	?	707·1	187·5?	189?	188·5?	130?	94	143	141	117	104	310	?
18	,,	♂	1595	537·4	174	175·5	175	153	96·5	130	129·5	115	96	318	320
19	,,	♂	1629·5	685·6	196·5	193·5	196	142·5	94·5	123·5	124	113	94	310	313
23	,,	♂	1775·5	691·5	200·5	199	200	139·5	102	143·5	143	121	113	325	324
24	,,	♂	1515·5	749·2	183	184	185	134·5	95·5	140	139·5	117	97·5	313	314
26	,,	♂	1498	647·3	175	175·5	174·5	143·5	93	132·5	131·5	116	95	318	321
27	,,	♂	1744	712·0	194·5	196	195·5	144	90	138·5	137·5	122	107	327	327
30	,,	♂	1549·5	698·4	192·5	197·5	197	131·5	96·5	135	134	111·5	102·5	300	302
31	,,	♂	1364	526·6	181·5	183·5	182·5	131	88	135	135	111·5	106	302	301

		Arcs								Face					
OH	*LB*	*Q*	*Q′*	*S*	S_1	S_2	S_3	S_3'	*U*	*PH*	*G′H*	*GB*	*J*	*NH R*	*NH L*
121	102	332	333	391	136·5	123	131·5	109	542	20·6	73·5	98·5	136·5	53·7	54·9
113	94	301	300	358	128·5	130	99·5	88	490	17·1	63	93·5	121	47·0	48·1
106·5	90·5	288	289	343·5	124·5	104	115	98·5	480	20·9?	69·5?	94·5	119·5	50·7	50·9
118	92	329	330	362	123·5	135	103·5	88·5	509	18·4	64·5	97	128	45·2	46·6
114	93	315	316	366·5	125	129·5	112	94	510	22·5	77·5	102	135·5	57·0	57·2
118	96·5	321	323	380·5	137·5	132	111	94	510	18·0	71·5	102	131	56·0	53·7
110·5	94·5	304	304	371·5	127	122	122·5	97·5	510	21·1	70·5	104	134·5	50·4	50·4
112·5	93	307	309	371·5	125	123·5	123	97	510	23·8	73·5	94	132·5	52·8	52·9
108·5	96·5	301	303	358	121·5	124	112·5	94	501	15·4	70	99·5	133·5	54·0	54·0
113·5	94·5	315	314	344	123	125	96	86·5	482	15·7	62	91·5	126	47·4?	47·4?
114·5	99·5	314	315	366·5	126	130	110·5	94·0	499	18·8	71	98·5	124	51·3	50·4
108·5	100	313	309	336	121	107	108	92	508	13·6	65	107	146·5	52·2	51·4
113	102·5	305	308	350·5	114·5	120	116	99	523	21·9	74·5	94·5	134·5	51·5	51·5
109	91	308	314	371·5	131	111·5	129	97	535	23·1?	76·5?	102	142	53·4	54·1
112·5	93·5	304	303	348	116	125	107	90·5	472	19·1	68	100·5	119	50·4	49·4
116	95	312	313	362	126·5	118	117·5	98	499	21·1	70·5	99·5	133·5	51·1	50·0
115·5	99	309	300	354·5	131	125	98·5	87	481	17·0	66	99	118·5	49·9	50·3
116·5	98	311	310	376	131·5	126·5	118	100·5	516	24·6	76	103	134	51·6	51·2
112	96	302	301	367·5	125	118	124·5	103·5	515	18·1	74	97	132	55·1	54·9
123·5	96·5	329	332	405·5	136	139	130·5	104	530	23·6	72·5	92·5	131·5	48·5	50·1
114·5	95	315	313	388	127	118	143	115·5	528	20·2	76·5	96·5	141·5	57·5	56·0
113·5	96	310	310	371·5	127	122	122·5	103·5	520	25·3	79	101·5	136·5	54·8	54·8
108	91·5	291	293	352·5	121·5	121·5	109·5	91	500	21·0	71·5	99	136·5	52·0	51·5
117	104	310	?	386·5?	124·5	142?	119·5	98	?	22·7	79·5	99	136	57·3	58·3
115	96	318	320	357	126	119	112	94	524	19·5	76·5	104·5	142	57·8	59·3
113	94	310	313	400	126	140	134	98·5	547	20·2	76	96·5	134	54·6	53·8
121	113	325	324	393	136·5	134·5	122	99	558	26·0	82·5	107	146	57·8	56·9
117	97·5	313	314	388	133	129·5	125·5	103·5	520	24·2	81·5	108	137	58·0	58·1
116	95	318	321	369·5	131·5	126	112	89	513	21·7	74	95·5	136·5	54·6	54·0
122	107	327	327	386·5	132	125	129·5	106	547	22·1	78·5	106	144	56·4	57·0
111·5	102·5	300	302	391·5	131·5	121	139	110·5	535	23·1	80·5	104·5	139	57·7	59·3
111·5	106	302	301	354·5	131	125	98·5	87·5	505	18·2	68·5	99·5	136	49·7	50·5

APPENDIX II. TABLE OF ABSOLUTE AND INDICIAL

Nose								Orbits					Palate and Profile				
NH R	*NH L*	*NB*	*DS*	*DC*	*DA*	*SS*	*SC*	O_1R	O_1L	O_2R	O_2L	O_1'	G_1	G_1'	G_2	*GL*	*EB*
53·7	54·9	24·2	6·2	22·4	25·5	2·2	8·1	41·5	42·6	36·9	37·1	39·5	48·3	45·7	40·0	93·5	40·0
47·0	48·1	23·9	7·9	21·1	26·0	2·6	8·0	40·2	39·7	34·3	34·6	39·1	43·5	40·9	36·0	93·0	36·3
50·7	50·9	26·3	7·4	18·4	22·0	1·8	7·2	38·8	38·8	37·1	36·6	36·5	48·7	45·1	39·1	89·5?	41·1
45·2	46·6	24·6	8·6	23·0	30·0	2·6	9·1	41·4	41·0	32·7	31·5	41·0	48·5	46·3	38·1	90·0	37·6
57·0	57·2	24·5	6·2	24·5	27·0	0·9	3·2	45·3	43·6	37·0	37·9	41·1	50·6	46·5	44·0	94·0	44·0
56·0	53·7	26·1	8·8	21·7	29·5	2·7	8·5	39·3	40·0	36·2	35·9	37·5	45·8	42·3	40·6	86·5	41·0
50·4	50·4	28·1	8·8	23·0	31·0	2·9	8·0	42·1	42·1	33·0	35·0	40·5	50·6	47·6	40·4?	90·5?	45·9?
52·8	52·9	29·9	7·6	20·9	27·5	1·5	4·8	44·0	43·8	35·2	36·3	41·5	48·7	44·8	45·0	89·0	45·1
54·0	54·0	25·4	6·8	25·9	29·5	2·4	8·9	42·0	41·7	37·1	36·9	39·6	48·1	45·2	45·0	85·0	44·3
47·4?	47·4?	24·2	?	21·9	?	?	?	41·6	40·7	31·4	31·5	37·8	46·3	43·8	36·2	94·5	36·1
51·3	50·4	24·7	11·5	22·5	33·0	4·2	10·4	40·1	40·6	35·3	36·0	38·8	48·2	45·1	42·4	91·5	42·3
52·2	51·4	28·0	9·2	22·2	29·0	2·5	7·8	43·4	43·5	34·0	33·7	39·9	47·0	43·9	44·2	94·5	44·2
51·5	51·5	24·4	9·5	19·9	26·5	3·0	6·5	40·4	41·7	34·9	33·8	39·1	52·4	48·9	44·4	105	44·4
53·4	54·1	25·2	8·2	23·0	29·0	2·9	8·8	43·3	42·3	36·0	35·5	40·4	49·7	45·5	44·3?	92·5?	45·1?
50·4	49·4	24·0	8·4	20·5	29·0	2·4	8·8	40·8	41·0	33·4	32·3	39·7	43·0	40·4	41·1	87·0	41·0
51·1	50·0	22·0	9·2	23·3	28·5	2·1	5·8	40·9	42·4	36·8	37·2	39·2	44·6	41·3	36·9	86·0	37·2
49·9	50·3	25·2	5·5	21·7	25·0	0·9	4·2	43·0	42·8	34·4	34·0	41·6	47·6	43·8	42·0	94·0	42·1
51·6	51·2	29·8	10·9	23·3	33·0	2·7	7·2	44·7	43·1	37·0	36·2	42·4	52·4	48·9	43·3	102	43·4
55·1	54·9	27·8	7·3	20·5	25·5	3·3	8·1	45·6	43·3	37·0	37·4	40·9	50·6	47·1	41·4	95	41·7
48·5	50·1	27·1	9·9	20·8	30·0	1·5	7·9	42·9	41·8	34·0	35·2	38·6	51·1	48·0	40·4	96·5	40·2
57·5	56·0	26·1	10·6	24·9	33·5	2·1	5·1	45·9	43·7	34·8	36·0	43·8	?	?	44·1	90·5	44·4
54·8	54·8	25·1	10·3	19·6	30·5	3·5	6·1	42·7	42·2	37·0	36·9	40·3	56·9	52·9	45·2	105·5	46·2
52·0	51·5	27·5	7·5	22·5	27·5	2·0	5·5	43·1	43·9	36·0	35·3	40·6	46·2	43·6	43·7	88	44·1
57·3	58·3	24·9	7·2	21·5	28·0	3·0	10·4	41·7	42·2	35·3	35·5	39·4	55·0	49·5	40·4	101	40·3
57·8	59·3	27·8	7·8	23·2	28·0	3·0	9·6	44·0	42·1	38·1	38·9	41·4	51·4	47·9	47·9	92·5	48·2
54·6	53·8	25·7	8·2	26·5	31·0	2·3	10·2	40·2	39·9	34·6	33·9	39·8	52·0	48·0	40·4	96·5	41·1
57·8	56·9	25·1	9·6	23·2	28·5	2·3	5·7	48·1	48·2	36·0	37·7	43·4	54·5	48·8	43·6	106	44·0
58·0	58·1	27·5	7·2	21·7	26·5	2·1	6·2	46·1	46·4	42·0	40·9	41·9	?	?	49·5	97	47·0
54·6	54·0	26·9	9·7	23·1	32·0	3·0	9·6	43·2	42·2	34·7	35·6	39·8	46·6	43·2	39·9	89·5	39·8
56·4	57·0	27·4	7·8	26·1	30·0	2·9	7·9	43·5	43·1	37·5	37·5	41·5	51·9	48·0	49·0	104	50·0
57·7	59·3	30·1	7·6	20·1	25·5	1·5	6·2	45·2	45·1	40·3	40·6	43·4	53·2	50·5	47·5?	98?	?
49·7	50·5	26·9	9·9	23·8	31·5	3·4	9·1	42·9	44·0	32·8	33·2	40·4	47·4	43·5	41·1	95·5	41·1

I. TABLE OF ABSOLUTE AND INDICIAL MEASUREMENTS OF THE TIBETA INDIVI

	Orbits					Palate and Profile						Foramen									
SC	O_1R	O_1L	O_2R	O_2L	O_1'	G_1	G_1'	G_2	*GL*	*EB*	*EH*	*fml*	*fmb*	*Oc.I*	*B/L'*	*H/L'*	*B/L*	*H/L*	*B/H*	*H'/L*	*B/H'*
8·1	41·5	42·6	36·9	37·1	39·5	48·3	45·7	40·0	93·5	40·0	12·4	40·8	34·2	59·6	79·6	76·0	80·1	76·4	105·2	75·6	105·9
8·0	40·2	39·7	34·3	34·6	39·1	43·5	40·9	36·0	93·0	36·3	11·7	34·2	31·8	67·8	76·1	72·3	76·5	72·8	105·2	73·3	104·3
7·2	38·8	38·8	37·1	36·6	36·5	48·7	45·1	39·1	89·5?	41·1	8·8	34·8	31·8	61·7	78·3	74·5	77·8	74·2	103·6	76·0	102·4
9·1	41·4	41·0	32·7	31·5	41·0	48·5	46·3	38·1	90·0	37·6	13·4	32·9	23·6	62·7	88·5	78·5	88·8	78·8	112·7	77·0	115·3
8·2	45·3	43·6	37·0	37·9	41·1	50·6	46·5	44·0	94·0	44·0	14·4	36·2	32·0	60·6	78·9	72·1	79·1	72·0	109·9	71·7	110·4
8·5	39·3	40·0	36·2	35·9	37·5	45·8	42·3	40·6	86·5	41·0	13·0	37·2	30·1	61·6	81·2	81·2	80·3	80·3	100·0	80·0	100·4
8·0	42·1	42·1	33·0	35·0	40·5	50·6	47·6	40·4?	90·5?	45·9?	7·8?	34·6	30·9	56·8	79·4	71·0	79·4	71·0	111·8	70·8	112·2
8·8	44·0	43·8	35·2	36·3	41·5	48·7	44·8	45·0	89·0	45·1	11·4	33·3	30·7	56·3	76·7	73·0	76·9	74·4	103·4	73·3	104·9
8·9	42·0	41·7	37·1	36·9	39·6	48·1	45·2	45·0	85·0	44·3	14·0	34·5	28·9	60·2	78·1	74·1	78·1	75·2	103·5	74·4	105·0
?	41·6	40·7	31·4	31·5	37·8	46·3	43·8	36·2	94·5	36·1	9·6	33·9	30·7	72·0	85·0	80·0	85·5	80·6	106·1	80·0	106·9
9·4	40·1	40·6	35·3	36·0	38·8	48·2	45·1	42·4	91·5	42·3	12·3	31·7	27·0	62·1	75·4	76·0	75·9	76·2	99·3	75·6	100·0
8·8	43·4	43·5	34·0	33·7	39·9	47·0	43·9	44·2	94·5	44·2	12·1	39·5	32·0	62·3	86·3	76·5	86·6	77·3	112·1	77·0	112·5
7·5	40·4	41·7	34·9	33·8	39·1	52·4	48·9	44·4	105	44·4	12·6	41·1	33·2	62·5	76·6	72·7	77·0	73·1	105·3	72·3	105·7
8·8	43·3	42·3	36·0	35·5	40·4	49·7	45·5	44·3?	92·5?	45·1?	12·0?	38·0	30·4	54·5	81·2	66·2	81·4	66·4	122·6	66·1	123·1
8·8	40·8	41·0	33·4	32·3	39·7	43·0	40·4	41·1	87·0	41·0	12·3	32·8	25·0	60·4	76·3	76·8	76·7	77·3	99·2	76·7	100·0
8·8	40·9	42·4	36·8	37·2	39·2	44·6	41·3	36·9	86·0	37·2	10·4	36·2	31·7	60·1	81·4	79·1	81·4	79·1	102·9	78·5	103·7
9·2	43·0	42·8	34·4	34·0	41·6	47·6	43·8	42·0	94·0	42·1	8·3	35·0	27·0	67·7	74·0	78·7	74·4	79·1	94·1	78·5	94·8
9·2	44·7	43·1	37·0	36·2	42·4	52·4	48·9	43·3	102	43·4	14·4	38·2	31·6	62·3	74·5	75·9	74·5	75·9	98·2	76·2	97·8
9·1	45·6	43·3	37·0	37·4	40·9	50·6	47·1	41·4	95	41·7	13·3	37·0	32·2	59·8	75·2	71·6	75·2	71·6	105·1	71·0	105·9
8·9	42·9	41·8	34·0	35·2	38·6	51·1	48·0	40·4	96·5	40·2	14·1	32·9	30·9	56·8	73·2	73·2	73·6	73·8	100·0	72·8	101·1
9·1	45·9	43·7	34·8	36·0	43·8	?	?	44·1	90·5	44·4	15·3	37·8	30·0	57·6	76·7	72·5	76·4	72·9	105·2	72·4	106·0
9·1	42·7	42·2	37·0	36·9	40·3	56·9	52·9	45·2	105·5	46·2	13·7	38·6	31·5	61·3	77·8	72·6	78·5	72·2	107·2	72·1	108·8
9·5	43·1	43·9	36·0	35·3	40·6	46·2	43·6	43·7	88	44·1	14·5	38·0	31·0	59·8	78·7	72·6	78·7	72·6	107·3	73·2	107·5
9·4	41·7	42·2	35·3	35·5	39·4	55·0	49·5	40·4	101	40·3	13·1	34·8	30·1	58·7	68·8?	75·7?	69·5?	75·8?	90·9	?	?
9·6	44·0	42·1	38·1	38·9	41·4	51·4	47·9	47·9	92·5	48·2	13·0	36·8	31·6	60·7	87·2	74·0	87·4	74·3	117·7	74·0	118·1
9·2	40·2	39·9	34·6	33·9	39·8	52·0	48·0	40·4	96·5	41·1	13·2	35·5	29·2	54·0	73·6	63·8	72·7	63·0	115·3	63·3	114·9
9·7	48·1	48·2	36·0	37·7	43·4	54·5	48·8	43·6	106	44·0	14·2	43·2	33·1	58·0	69·9	72·1	69·8	71·8	97·2	71·5	97·6
9·2	46·1	46·4	42·0	40·9	41·9	?	?	49·5	97	47·0	17·8	35·5	30·8	59·1	72·1	76·1	72·7	75·7	96·1	75·4	96·4
9·6	43·2	42·2	34·7	35·6	39·8	46·6	43·2	39·9	89·5	39·8	11·4	32·6	30·6	56·7	81·8	75·5	82·1	75·9	108·3	75·4	109·1
9·9	43·5	43·1	37·5	37·5	41·5	51·9	48·0	49·0	104	50·0	12·3	42·3	35·5	58·5	73·5	70·7	73·7	70·8	104·0	70·3	104·7
9·2	45·2	45·1	40·3	40·6	43·4	53·2	50·5	47·5?	98?	?	?	40·2	33·9	56·7	66·6	68·4	66·8	68·5	97·4	68·0	98·1
9·1	42·9	44·0	32·8	33·2	40·4	47·4	43·5	41·1	95·5	41·1	13·3	37·7	28·5	68·7	71·4	73·6	71·8	74·0	97·0	74·0	97·0

INDIVIDUAL CRANIA (LENGTHS IN MILLIMETRES).

Indices

	H'/L	B/H'	$\frac{(B-H')}{L}$	$G'H/GB$	$\frac{NB/NH}{R}$	$\frac{NB/NH}{L}$	$\frac{O_2/O_1}{R}$	$\frac{O_2/O_1}{L}$	O_2/O_1'	G_2/G_1	EH/EB	fmb/fml	DS/DC	SS/SC	$N\angle$	$A\angle$
2	75·6	105·9	4·5	74·6	45·1	44·1	88·9	87·1	93·4	82·8	31·0	83·8	27·7	27·2	1°·7	74°·6
2	73·3	104·3	3·2	67·4	50·9	49·7	85·3	87·2	87·7	82·8	32·2	93·0	37·4	32·5	9°·4	71°·3
6	76·0	102·4	1·8	?	51·9	51·7	95·6	94·3	101·6	80·3	21·4	91·4	40·2	25·0	5°·5?	68°·9?
7	77·0	115·3	11·8	66·5	54·4	52·8	79·0	76·8	79·8	78·6	35·6	71·7	37·4	28·6	7°·3	71°·2
9	71·7	110·4	7·4	76·0	43·0	42·8	81·7	86·9	90·0	87·0	32·7	88·4	25·3	28·1	6°·1	65°·0
0	80·0	100·4	0·3	70·1	46·6	48·6	92·1	89·8	96·5	88·6	31·7	80·9	40·6	31·8	9°·8	74°·9
8	70·8	112·2	8·6	67·8	55·8	55·8	78·4	83·1	81·5	?	17·0?	89·3	38·3	36·3	4°·6	70°·8
4	73·3	104·9	3·6	78·2	56·6	56·5	80·0	82·9	84·8	92·4	25·3	92·2	36·4	31·2	3°·2	69°·4
5	74·4	105·0	3·7	70·4	47·0	47·0	88·3	88·5	93·7	93·6	31·6	83·8	26·3	27·0	8°·8	76°·6
1	80·0	106·9	5·5	67·8	51·1?	51·1?	75·5	77·4	83·1	78·0	26·6	90·6	?	?	0°·8	71°·1
3	75·6	100·0	0·3	72·2	48·1	49·0	88·0	88·7	91·0	88·0	29·1	85·2	51·1	40·4	2°·1	74°·6
1	77·0	112·5	9·6	60·8	53·6	54·5	78·3	78·2	85·2	94·0	27·4	81·0	41·4	32·1	5°·8	75°·4
3	72·3	105·7	4·7	78·8	47·4	47·4	86·4	81·1	89·3	84·7	28·4	80·8	47·7	46·1	0°·9	67°·2
6	66·1	123·1	15·3	?	47·2	46·6	83·1	83·9	89·1	?	26·6?	80·0	35·7	32·9	6°·2	64°·7
2	76·7	100·0	0·0	67·7	47·6	48·6	81·9	78·8	84·1	95·6	30·0	76·2	41·0	27·3	2°·9	73°·3
9	78·5	103·7	2·9	70·9	43·1	44·0	90·0	87·7	93·9	82·7	28·0	87·6	39·5	36·2	0°·1	73°·9
1	78·5	94·8	−4·1	66·7	50·5	50·1	80·0	79·4	82·7	88·2	19·7	77·1	25·3	21·4	5°·9	74°·3
2	76·2	97·8	−1·7	73·8	57·8	58·2	82·8	84·0	87·3	82·6	32·7	82·7	46·8	37·5	0°·3	65°·2
1	71·0	105·9	4·2	76·3	50·5	50·6	82·0	86·4	90·5	81·8	31·9	87·0	35·6	40·8	6°·3	68°·2
0	72·8	101·1	0·8	78·4	55·9	54·1	79·3	84·2	88·1	79·1	35·1	93·9	47·6	19·0	7°·9	68°·0
2	72·4	106·0	4·0	79·3	45·4	46·6	75·8	82·4	79·5	?	34·5	79·4	42·6	41·2	2°·5	69°·0
2	72·1	108·8	6·4	78·3	45·8	45·8	86·7	87·4	91·8	79·4	29·6	81·6	52·6	57·4	3°·3	61°·0
3	73·2	107·5	5·5	72·2	52·9	53·4	83·5	80·4	88·7	94·6	32·9	81·6	33·3	36·4	3°·9	69°·3
9	?	?	?	80·3	43·5	42·7	84·7	84·1	89·6	73·5	32·5	86·5	33·5	28·8	5°·0	69°·6
7	74·0	118·1	13·4	73·2	48·1	46·9	86·6	92·4	92·0	93·2	27·0	85·9	37·9	31·3	3°·6	68°·6
3	63·3	114·9	9·4	78·8	47·1	47·8	86·1	85·0	86·9	77·7	32·1	82·3	30·9	22·5	8°·1	65°·0
2	71·5	97·6	−1·7	77·1	43·4	44·1	74·8	78·2	82·9	80·0	32·3	76·6	41·4	40·3	3°·3	72°·8
1	75·4	96·4	−2·7	74·5	47·4	47·3	91·1	88·1	100·2	?	37·9	86·8	33·2	33·9	4°·9	65°·7
3	75·4	109·1	6·7	78·3	49·3	49·8	80·3	84·4	87·2	85·6	28·6	93·9	42·0	31·3	2°·2	70°·6
0	70·3	104·7	3·4	74·1	48·6	48·1	86·2	87·0	90·4	94·4	24·6	83·9	29·9	36·7	6°·0	70°·4
4	68·0	98·1	−1·2	77·0	52·2	50·7	89·2	90·0	92·9	?	?	84·3	37·8	24·2	3°·2	69°·5
0	74·0	97·0	2·2	68·8	54·1	53·3	76·5	75·5	81·2	86·7	32·4	75·6	41·6	37·4	1°·8	78°·9

IN MILLIMETRES).

O_2/O_1 R	O_2/O_1 L	O_2/O_1'	G_2/G_1	EH/EB	fmb/fml	DS/DC	SS/SC	Angles $N\angle$	$A\angle$	$B\angle$	θ_1	θ_2	$P\angle$	
88·9	87·1	93·4	82·8	31·0	83·8	27·7	27·2	51°·7	74°·6	43°·7	31°·8	11°·9	86°·5	Young adult; 3rd molars in lower j and ossicle at lambda; ossicles at
85·3	87·2	87·7	82·8	32·2	93·0	37·4	32·5	69°·4	71°·3	39°·3	25°·6	13°·7	85°·0	Young; 3rd molars beginning to er foration R. and L.; epipteric bon
95·6	94·3	101·6	80·3	21·4	91·4	40·2	25·0	65°·5?	68°·9?	45°·6?	31°·0	14°·6?	83°·5?	Ageing; no 3rd molars; teeth worn wormians lambdoid; two distinct
79·0	76·8	79·8	78·6	35·6	71·7	37·4	28·6	67°·3	71°·2	41°·5	24°·7	18°·8	90°·0	Young; 3rd molars both jaws just wormians lambdoid.
81·7	86·9	90·0	87·0	32·7	88·4	25·3	28·1	66°·1	65°·0	48°·9	26°·9	22°·0	87°·0	P.B. sloping upwards and outwards R. and L.; epipteric bones R. an
92·1	89·8	96·5	88·6	31·7	80·9	40·6	31·8	59°·8	74°·9	45°·3	32°·2	13°·1	88°·0	P.B. flat with blunt edge; S equal;
78·4	83·1	81·5	?	17·0?	89·3	38·3	36·3	64°·6	70°·8	44°·6	26°·4	18°·2	89°·0	Ageing; alveolus partly absorbed; outwards with sharp edge; SR; J
80·0	82·9	84·8	92·4	25·3	92·2	36·4	31·2	63°·2	69°·4	47°·4	29°·8	17°·6	87°·0	One diseased molar; P.B. flat with small epipteric bones L.
88·3	88·5	93·7	93·6	31·6	83·8	26·3	27·0	58°·8	76°·6	64°·6	30°·7	13°·9	90°·5	Young; upper molars erupting but nasal spine; P.B. sloping upwar and L.; epipteric bone L.; proces
75·5	77·4	83·1	78·0	26·6	90·6	?	?	70°·8	71°·1	38°·1	26°·2	11°·9	83°·0	Protruding L. upper canine; P.B. fl
88·0	88·7	91·0	88·0	29·1	85·2	51·1	40·4	62°·1	74°·6	43°·3	30°·4	12°·9	87°·5	No 3rd molars upper jaw; P.B. slo
78·3	78·2	85·2	94·0	27·4	81·0	41·4	32·1	65°·8	75°·4	38°·8	28°·2	10°·6	86°·0	No 3rd molars either jaw; P.B. fla R. and L.
86·4	81·1	89·3	84·7	28·4	80·8	47·7	46·1	70°·9	67°·2	41°·9	27°·6	14°·3	81°·5	P.B. flat with sharp edge; SR; JR
83·1	83·9	89·1	?	26·6?	80·0	35·7	32·9	66°·2	64°·7	49°·1	24°·3?	24°·8?	89°·5?	Ageing; sagittal suture beginning lambdoid; slight palatine torus;
81·9	78·8	84·1	95·6	30·0	76·2	41·0	27·3	62°·9	73°·3	43°·8	26°·1	17°·7	91°·0	Supernumerary tooth between alve JR; wormians lambdoid; epipter
90·0	87·7	93·9	82·7	28·0	87·6	39·5	36·2	60°·1	73°·9	46°·0	31°·9	14°·1	88°·0	Old; coronal and sagittal sutures cl
80·0	79·4	82·7	88·2	19·7	77·1	25·3	21·4	65°·9	74°·3	39°·8	23°·1	16°·7	91°·0	P.B. sloping upwards and outwards
82·8	84·0	87·3	82·6	32·7	82·7	46·8	37·5	70°·3	65°·2	44°·5	30°·7	13°·8	79°·0	Sagittal suture closing at obelion; P.
82·0	86·4	90·5	81·8	31·9	87·0	35·6	40·8	66°·3	68°·2	45°·5	27°·2	18°·3	86°·5	Ageing; sagittal suture completely sharp edge; S equal; JR; slight
79·3	84·2	88·1	79·1	35·1	93·9	47·6	19·0	67°·9	68°·0	44°·1	25°·6	18°·5	86°·5	P.B. flat with blunt edge; SR; JR
75·8	82·4	79·5	?	34·5	79·4	42·6	41·2	62°·5	69°·0	48°·5	31°·5	17°·0	86°·0	Sagittal suture closing at obelion a metopic; wormians lambdoid; ty
86·7	87·4	91·8	79·4	29·6	81·6	52·6	57·4	73°·3	61°·0	45°·7	28°·7	17°·0	78°·0	P.B. flat with blunt edge; SL; JR;
83·5	80·4	88·7	94·6	32·9	81·6	33·3	36·4	63°·9	69°·3	46°·8	31°·1	15°·7	85°·0	P.B. flat with blunt edge; SR; JR
84·7	84·1	89·6	73·5	32·5	86·5	33·5	28·8	65°·0	69°·6	45°·4	30°·5	14°·9	84°·5	Very protruding upper incisives; i pre-condyles; prognathous; calv
86·6	92·4	92·0	93·2	27·0	85·9	37·9	31·3	63°·6	68°·6	47°·8	27°·4	20°·4	89°·0	Small 3rd molars; P.B. flat with b
86·1	85·0	86·9	77·7	32·1	82·3	30·9	22·5	68°·1	65°·0	46°·9	23°·4	23°·5	88°·5	Diseased molar; small protruding perforation R. and L.; bipartite
74·8	78·2	82·9	80·0	32·3	76·6	41·4	40·3	63°·3	72°·8	43°·9	26°·7	17°·2	90°·0	P.B. sloping upwards and outward bones R.
91·1	88·1	100·2	?	37·9	86·8	33·2	33·9	64°·9	65°·7	49°·4	32°·1	17°·3	83°·0	4 diseased teeth; P.B. sloping upw
80·3	84·4	87·2	85·6	28·6	93·9	42·0	31·3	62°·2	70°·6	47°·2	28°·3	18°·9	89°·5	No 3rd molars; P.B. flat with dou
86·2	87·0	90·4	94·4	24·6	83·9	29·9	36·7	66°·0	70°·4	43°·6	26°·5	17°·1	87°·5	4 diseased teeth; P.B. flat with bl
89·2	90·0	92·9	?	?	84·3	37·8	24·2	63°·2	69°·5	47°·3	29°·8	17°·5	87°·0	Ageing; sagittal suture closing at S equal; J equal; wormians lam
76·5	75·5	81·2	86·7	32·4	75·6	41·6	37·4	61°·8	78°·9	39°·3	29°·7?	9°·6?	88°·5?	Sagittal suture closing at vertex; epipteric bones R. and L.

Remarks*
molars in lower jaw only; P.B. sloping upwards and outwards with sharp edge; SR; JL; wormians R. and L. lambdoid
mbda; ossicles at asterion R. and L.; epipteric bones R. and L.
s beginning to erupt in mandible; infantile type of forehead; P.B. flat with sharp edge; S equal; JR; tympanic per-
L.; epipteric bones L.
olars; teeth worn and 1st molars of mandible lost; P.B. sloping upwards and outwards with sharp edge; SR; JR;
loid; two distinct pre-condyles; tympanic perforation R. and L.; short projecting spine at opisthion.
rs both jaws just erupting; P.B. flat with double, sharp edges and groove between; SR; JR; slight asymmetry;
loid.
rds and outwards with sharp edge; SR; JR; metopic; wormians lambdoid and ossicle L. coronal; ossicles at asterion
teric bones R. and L.; flattening at obelion.
nt edge; S equal; JR; epipteric bones R. and L.
partly absorbed; remaining teeth worn; sagittal suture closing at obelion; no 3rd molars; P.B. sloping upwards and
harp edge; SR; JR; wormians lambdoid; two distinct pre-condyles; very low, retreating forehead.
r; P.B. flat with rounded edge; SR; JL; metopic; wormians lambdoid and ossicle of lambda; ossicles at asterion R.;
ones L.
ars erupting but lower ones fully developed; tooth appearing in floor of pyriform aperture immediately behind anterior
B. sloping upwards and outwards with sharp edge; SR; JR; metopic; wormians lambdoid; ossicles at asterion R.
e bone L.; process between temporal and parietal bones R. and L.
er canine; P.B. flat with blunt edge; J equal; S equal; slight asymmetry; ossicle L. lambdoid; tympanic perforation L.
per jaw; P.B. sloping upwards and outwards with sharp edge; S equal; JR; slight asymmetry.
her jaw; P.B. flat with blunt edge; teeth worn; SR; JR; wormians lambdoid and ossicle of lambda; epipteric bones
rp edge; SR; JR; slight asymmetry; ossicle R. coronal and wormians R. lambdoid; tympanic perforation R. and L.
uture beginning to close at obelion; alveolus partly absorbed; P.B. flat with blunt edge; S equal; JR; wormians
palatine torus; ossicles at asterion R. and L.; bipartite interparietal; epipteric bones L.
oth between alveolar borders of 2nd and 3rd incisives; P.B. flat with double, sharp edges and groove between; S equal;
ambdoid; epipteric bone L.
agittal sutures closed at bregma; P.B. sloping upwards and outwards with sharp edge; SR; JR.
rds and outwards with sharp edge; SR; JR; epipteric bones R. and L.
sing at obelion; P.B. sloping upwards and outwards with blunt edge; SR; JR; prognathous; ossicles at asterion R. and L.
uture completely closed, other sutures beginning to close; 1 molar lost; P.B. sloping upwards and outwards with
qual; JR; slight asymmetry.
nt edge; SR; JR; slight asymmetry; slight metopic ridge; ossicles lambdoid.
osing at obelion and coronal at extremities R. and L.; teeth worn and 3 lost; P.B. flat with sharp edge; S equal; JR;
ans lambdoid; tympanic perforation R. and L.
nt edge; SL; JR; ossicle R. lambdoid; epipteric bones R. and L.
nt edge; SR; JR; tympanic perforation R. and L.; epipteric bone L.
upper incisives; inner palate bridge L.; P.B. sloping upwards and outwards with rounded edge; S equal; JR; fused
rognathous; calvaria damaged by shot.
P.B. flat with blunt edge; S equal; JR; wormians lambdoid; tuberculum pharyngeum replaced by a depression.
mall protruding bud on mesial surface of upper 3rd molar; P.B. flat with blunt edge; S equal; JL; metopic; tympanic
nd L.; bipartite interparietal.
ards and outwards with sharp edge; SR; JL; very asymmetric; metopic; wormians coronal and lambdoid; epipteric
P.B. sloping upwards and outwards with sharp edge; S equal; JR; flattening at obelion; small epipteric bone R.
.B. flat with double, sharp edges; SR; JR.
P.B. flat with blunt edge; SR; JR; wormians lambdoid; tympanic perforation R. and L.; epipteric bone L.
uture closing at obelion; 1 remaining tooth in upper jaw but 11 in mandible; teeth worn; P.B. flat with sharp edge;
l; wormians lambdoid; tympanic perforation R. and L.; simple interparietal.
osing at vertex; teeth worn; P.B. sloping upwards and outwards with sharp edge; SR; JR; wormians R. lambdoid;
R. and L.

* See Section 4 of Text for meaning of letters etc. used.

THE PROBABLE ERROR OF A CLASS-INDEX CORRELATION.

By EGON S. PEARSON, B.A.

(1) The correlation between the quantitative value of a variate and its class-index when the variate is classed or "indexed" in broad categories is termed the "class-index correlation." As a rule this class-index is taken as the value of the mean of the variates falling within the category, but it may also be taken as their median, or as the mid-point of the sub-range corresponding to the broad category. In this paper the class-index adopted is the mean of the variates falling into the broad category. The importance of the class-index correlation arises from its use as a correction for correlations determined from broad-category classifications, whether these are treated by contingency or by correlation-ratio methods. It has accordingly become a problem of very real interest to determine the probable error of a class-index correlation. Is such a correlation subject to a large error, so that it cannot reasonably be used as a correction? The object of this paper is to answer this question.

In previous papers* the probable errors have been given of various constants required in the description of the frequency distribution in a sample taken from a larger population. If the particular character in the population under consideration is classified only by broad qualitative categories, it is generally most convenient to fit the data to a Normal Scale, and the following notation is that in common use.

M is the total population supposed "indefinitely large," from which a sample N, small compared to M, is taken. On the other hand we are not dealing with "small samples" and shall neglect second order terms with factors $1/N^2$‡.

$$y_s = \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x_s^2}{\sigma^2}}, \qquad n_s = \int_{x_s}^{\infty} \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx,$$

where σ represents the standard deviation, n_s the frequency beyond x_s in a *sample*, and x is measured from the mean of the sample. As the data are divided into qualitative categories only, we do not know σ, y_s and x_s but only n_s†, and thence from Sheppard's Tables of the Normal Curve can find

$$h_s = \frac{x_s}{\sigma}; \quad H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2}; \quad \text{and } h_{r,s} = \frac{x_{r,s}}{\sigma} = \frac{H_s - H_r}{(n_s - n_r)/N},$$

where $x_{r,s}$ is the abscissa of the centroid of the class or group lying between x_s and x_r. For the *total population* we shall have corresponding constants $\tilde{h}_s$, $\tilde{h}_{r,s}$, $\tilde{H}_s$ and $\tilde{n}_s$, where

$$h_s = \tilde{h}_s + \delta h_s, \; h_{r,s} = \tilde{h}_{r,s} + \delta h_{r,s}, \; H_s = \tilde{H}_s + \delta H_s, \; n_s = \tilde{n}_s N/M + \delta n_s.$$

* *Biometrika*, Vol. II. pp. 273—281; Vol. IX. pp. 1—10; Vol. XIII. pp. 113—132.

† n_s/N is of course the $\frac{1}{2}(1-\alpha)$ of Sheppard's Tables.

‡ This is what a member of the Biometric School understands when he replaces "statistical differentials" by "mathematical differentials."

Now it has been shown in the earlier papers referred to in the footnote on p. 261, that if statistical differentials may be treated as mathematical differentials, or in other words $1/\sqrt{N}$ is small compared to unity, the following fundamental relations hold for the variations, δn_s and δn_r, in a sample of N from an indefinitely larger population:

$$\text{Mean}\,(\delta n_s^2) = \sigma^2_{n_s} = \frac{N\tilde{n}_s}{M}\left(1 - \frac{\tilde{n}_s}{M}\right)$$

$$= n_s\left(1 - \frac{n_s}{N}\right) \text{ approximately} \quad \ldots\ldots\ldots\ldots\text{(i)},$$

$$\text{Mean}\,(\delta n_s \delta n_r) = \frac{N\tilde{n}_r}{M}\left(1 - \frac{\tilde{n}_s}{M}\right)$$

$$= n_r\left(1 - \frac{n_s}{N}\right) \text{approximately} \quad \ldots\ldots\ldots\ldots\text{(ii)},$$

where in the second equation the frequency n_s includes n_r.

Starting from these two equations it is proposed in the present paper to find the probable errors due to random sampling when the sampled population is approximately normal,

(*a*) of $h_{r,s} = \frac{x_{r,s}}{\sigma}$, or the distance, in terms of the standard deviation as unit, of the mean of a group from the mean of the whole sample;

(*b*) of $r_{x.c_x}$, the Class-Index Correction* or correlation of a variate with its class-mark given by the relation

$$r^2_{x.\,c_x} = \frac{\sigma^2_{c_x}}{\sigma^2} = S\left\{\frac{n_s - n_{s+1}}{N} h^2_{s+1,\,s}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

where σ_{c_x} is the standard deviation of the means of the groups or class-marks, and the summation is for all groups.

(2) As a preliminary step it is necessary to find the following mean values of the squares and products of the variations of the constants in samples of N:

$\text{Mean}\,(\delta H_s^2) = \sigma^2_{H_s}$; $\text{Mean}\,(\delta H_s \delta H_r)$; $\text{Mean}\,(\delta H_s \delta n_r)$ and $\text{Mean}\,(\delta H_s \delta n_s)$, where n_s includes n_r.

Now
$$\tilde{n}_s = \int_{\tilde{h}_s}^{\infty} \frac{M}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2}\, dh,$$

and it follows, if we may treat statistical differentials as mathematical, that

$$\delta n_s = -\frac{N}{M}\frac{M}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{h}_s^2}\delta h_s = -N\tilde{H}_s \delta h_s,$$

δn_s being a variation in a sample of size N;

$$\delta H_s = -\frac{\tilde{h}_s}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\tilde{h}_s^2}\delta h_s = -\tilde{h}_s \tilde{H}\delta h_s,$$

and therefore
$$\delta H_s = \frac{1}{N}\tilde{h}_s \delta n_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

* *Biometrika*, Vol. IX. pp. 116—139.

Making use of (i), (ii) and (iv), it is found at once that

$$\text{Mean}\,(\delta H_s^2) = \sigma^2_{H_s} = \frac{\tilde{h}_s^2}{N}\frac{\tilde{n}_s}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....(v)},$$

$$\text{Mean}\,(\delta H_s \delta H_r) = \frac{\tilde{h}_s \tilde{h}_r}{N}\frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....(vi)},$$

$$\text{Mean}\,(\delta H_s \delta n_r) = \tilde{h}_s \frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....(vii)},$$

$$\text{Mean}\,(\delta H_r \delta n_s) = \tilde{h}_r \frac{\tilde{n}_r}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....(viii)},$$

$$\text{Mean}\,(\delta H_s \delta n_s) = \tilde{h}_s \frac{\tilde{n}_s}{M}\left(1-\frac{\tilde{n}_s}{M}\right) \quad \text{.....(ix)},$$

where in all cases n_s includes n_r.

The values of the constants in the sampled population are generally unknown, and for practical purposes it is therefore necessary to put into the above equations the values found from the sample*, or h_s, h_r, $\frac{n_s}{N}$ and $\frac{n_r}{N}$.

(3) $$\tilde{h}_{r,s} = \frac{\tilde{H}_s - \tilde{H}_r}{(\tilde{n}_s - \tilde{n}_r)/M},$$

and hence

$$\delta h_{r,s} = \frac{M}{\tilde{n}_s - \tilde{n}_r}(\delta H_s - \delta H_r) - \frac{M(\tilde{H}_s - \tilde{H}_r)}{(\tilde{n}_s - \tilde{n}_r)^2}(\delta n_s - \delta n_r)\frac{M}{N}$$

δn_s and δn_r being variations in a sample of size N,

$$= \tilde{h}_{r,s}\frac{\delta H_s - \delta H_r}{\tilde{H}_s - \tilde{H}_r} - \frac{\tilde{h}^2_{r,s}}{N}\frac{\delta n_s - \delta n_r}{\tilde{H}_s - \tilde{H}_r}.$$

After forming the equation of differentials, we may substitute $h_{r,s}$ for $\tilde{h}_{r,s}$ and H_s for $\tilde{H}_s$, etc., in this equation; then squaring both sides and taking the mean for all possible variations in random samples, we have

$$\text{Mean}\,(\delta h^2_{r,s}) = \frac{h^2_{r,s}}{(H_s - H_r)^2}\Big\{\text{Mean}\,(\delta H_s^2) + \text{Mean}\,(\delta H_r^2) - 2\times\text{Mean}\,(\delta H_s \delta H_r)$$
$$+ \frac{h^2_{r,s}}{N^2}(\text{Mean}\,(\delta n_s^2) + \text{Mean}\,(\delta n_r^2) - 2\times\text{Mean}\,(\delta n_s \delta n_r))$$
$$- 2\frac{h_{r,s}}{N}(\text{Mean}\,(\delta H_s \delta n_s) + \text{Mean}\,(\delta H_r \delta n_r) - \text{Mean}\,(\delta H_s \delta n_r) - \text{Mean}\,(\delta H_r \delta n_s))\Big\}.$$

Making use of the relations (v)—(ix) modified by the substitution of the

* For example $n_s = \frac{N\tilde{n}_s}{M} \pm \lambda\sigma_{n_s}$ where λ will on the average be greater than 2 in 4·5 °/₀ of samples, and greater than 3 in only about 0·3°/₀. Hence if the substitution of $\frac{n_s}{N}$ for $\frac{\tilde{n}_s}{M}$ is to be justifiable, $\frac{\lambda\sigma_{n_s}}{N}$ must be small compared to $\frac{\tilde{n}_s}{M}$ or $\frac{n_s}{N}$. Using Equation (i) it is found that this condition implies that $\lambda^2\left(1-\frac{n_s}{N}\right)$ should be small compared with n_s. $1-\frac{n_s}{N}$ is always less than unity, so that we see that in arranging our material, it is important to avoid groups containing less than 10 to 20 individuals; smaller cell contents should be combined together, or else the methods of "small samples" must be resorted to.

sample instead of the population values of the constants, and collecting terms, we find that

$$\sigma^2_{h_{r,s}} = \text{Mean}\,(\delta h^2_{r,s})$$
$$= \frac{N^2}{(n_s - n_r)^2}\left\{\frac{n_s}{N}\left(1-\frac{n_s}{N}\right)\left(\frac{h_s^2}{N}+\frac{h^2_{r,s}}{N}-2\frac{h_{r,s}h_s}{N}\right)+\frac{n_r}{N}\left(1-\frac{n_r}{N}\right)\left(\frac{h_r^2}{N}+\frac{h^2_{r,s}}{N}-2\frac{h_{r,s}h_r}{N}\right)\right.$$
$$\left.+2\frac{n_r}{N}\left(1-\frac{n_s}{N}\right)\left(-\frac{h_sh_r}{N}-\frac{h^2_{r,s}}{N}+\frac{h_{r,s}h_s}{N}+\frac{h_{r,s}h_r}{N}\right)\right\},$$

$$\sigma_{h_{r,s}} = \frac{\sqrt{N}}{n_s-n_r}\left\{\frac{n_s}{N}\left(1-\frac{n_s}{N}\right)(h_s-h_{r,s})^2+\frac{n_r}{N}\left(1-\frac{n_r}{N}\right)(h_r-h_{r,s})^2\right.$$
$$\left.-2\frac{n_r}{N}\left(1-\frac{n_s}{N}\right)(h_s-h_{r,s})(h_r-h_{r,s})\right\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots(x).$$

If the sample be large, no great error will be involved in assuming that the variations, $\delta h_{r,s}$, follow a normal distribution, so that we may take $\pm\cdot67449\,\sigma_{h_{r,s}}$ as the probable error of $h_{r,s}$.

(4) Let us suppose that our material is divided into p categories. Then by definition $\tilde{h}_0 = -\infty$, $\tilde{h}_p = +\infty$, $\tilde{H}_0 = 0 = \tilde{H}_p$, $\tilde{n}_0 = M$, $\tilde{n}_p = 0$. Further since

$$\tilde{h}_{s+1,\,s} = \frac{\tilde{H}_s - \tilde{H}_{s+1}}{(\tilde{n}_s - \tilde{n}_{s+1})/M},$$

we shall have for the means of the end categories

$$\tilde{h}_{10} = \frac{-\tilde{H}_1}{1-\frac{\tilde{n}_1}{M}}, \qquad \tilde{h}_{p,\,p-1} = \frac{\tilde{H}_{p-1}}{\tilde{n}_{p-1}/M}.$$

Now
$$\tilde{r}^2_{x.c_x} = \sum_{s=0}^{p-1}\left\{\frac{\tilde{n}_s - \tilde{n}_{s+1}}{M}\,\tilde{h}^2_{s+1,\,s}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{iii})\,bis$$
$$= M\sum_{s=0}^{p-1}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})^2}{\tilde{n}_s - \tilde{n}_{s+1}}\right\},$$

where it must be remembered that terms in $\tilde{H}_0$ and $\tilde{H}_p$ do not occur. Now if we treat statistical differentials as mathematical differentials, we have for the equation connecting the variations in $r_{x.c_x}$ with the variations in the $\tilde{H}$'s and $\tilde{n}$'s,

$$2\tilde{r}_{x.c_x}\,\delta r_{x.c_x} = 2M\sum_{s=0}^{p-1}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})(\delta H_s - \delta H_{s+1})}{\tilde{n}_s - \tilde{n}_{s+1}}\right\}$$
$$-M\sum_{s=0}^{p-1}\left\{\frac{(\tilde{H}_s - \tilde{H}_{s+1})^2\frac{M}{N}(\delta n_s - \delta n_{s+1})}{(\tilde{n}_s - \tilde{n}_{s+1})^2}\right\},$$

and on replacing $\tilde{H}_s$ by H_s, $\frac{\tilde{n}_s}{M}$ by $\frac{n_s}{N}$, etc., as we may do after differentiation, we find that

$$r_{x.c_x}\,\delta r_{x.c_x} = \sum_{s=1}^{p-1}\{h_{s+1,\,s}(\delta H_s - \delta H_{s+1})\} - \tfrac{1}{2}\sum_{s=0}^{p-1}\left\{\frac{h^2_{s+1,\,s}}{N}(\delta n_s - \delta n_{s+1})\right\}$$
$$= \sum_{s=1}^{p-1}\{\delta H_s(h_{s+1,\,s} - h_{s,\,s-1})\} - \tfrac{1}{2}\sum_{s=1}^{p-1}\left\{\frac{\delta n_s}{N}(h^2_{s+1,\,s} - h^2_{s,\,s-1})\right\},$$

since there are no terms in δH_0 and δH_p.

Squaring both sides of this equation, we find

$$r^2_{x.c_x}(\delta r_{x.c_x})^2 = \sum_{s=1}^{p-1} S \Big\{ \delta H_s^2 (h_{s+1,\,s} - h_{s,\,s-1})^2 + \frac{1}{4N^2} \delta n_s^2 (h^2_{s+1,\,s} - h^2_{s,\,s-1})^2 - \frac{1}{N} \delta H_s \delta n_s (h_{s+1,\,s} - h_{s,\,s-1})(h^2_{s+1,\,s} - h^2_{s,\,s-1}) \Big\}$$
$$+ 2 \sum_{j,k}{}' S \{ \delta H_j \delta H_k (h_{j+1,\,j} - h_{j,\,j-1})(h_{k+1,\,k} - h_{k,\,k-1}) \}$$
$$+ \frac{2}{4N^2} \sum_{j,k}{}' S \{ \delta n_j \delta n_k (h^2_{j+1,\,j} - h^2_{j,\,j-1})(h^2_{k+1,\,k} - h^2_{k,\,k-1}) \}$$
$$- \frac{1}{N} \sum_{j,k}{}' S \{ \delta H_j \delta n_k (h_{j+1,\,j} - h_{j,\,j-1})(h^2_{k+1,\,k} - h^2_{k,\,k-1}) \}$$
$$- \frac{1}{N} \sum_{j,k}{}' S \{ \delta H_k \delta n_j (h_{k+1,\,k} - h_{k,\,k-1})(h^2_{j+1,\,j} - h^2_{j,\,j-1}) \},$$

where $\sum_{j,k}{}' S$ indicates the summation for all pairs of integers (j not equal to k) between 1 and $p-1$. Now if we sum both sides of the last equation for all possible variations of $r_{x.c_x}$, of the H's and of the n's, and take the mean value, we have on making use of relations (iv) to (ix), and then collecting terms,

$$r^2_{x.c_x} \text{ Mean } [(\delta r_{x.c_x})^2] = r^2_{x.c_x} \sigma^2_{r_{x.c_x}}$$
$$= \sum_{s=1}^{p-1} S \Big\{ \frac{n_s}{N}\Big(1 - \frac{n_s}{N}\Big) \Big[(h_{s+1,\,s} - h_{s,\,s-1})^2 \frac{h_s^2}{N} + (h^2_{s+1,\,s} - h^2_{s,\,s-1})^2 \frac{1}{4N} - (h_{s+1,\,s} - h_{s,\,s-1})(h^2_{s+1,\,s} - h^2_{s,\,s-1}) \frac{h_s}{N} \Big] \Big\}$$
$$+ 2 \sum_{j,k}{}' S \Big\{ \frac{n_k}{N}\Big(1 - \frac{n_j}{N}\Big) \Big[(h_{j+1,\,j} - h_{j,\,j-1})(h_{k+1,\,k} - h_{k,\,k-1}) \frac{h_j h_k}{N}$$
$$+ (h^2_{j+1,\,j} - h^2_{j,\,j-1})(h^2_{k+1,\,k} - h^2_{k,\,k-1}) \frac{1}{4N}$$
$$- (h_{j+1,\,j} - h_{j,\,j-1})(h^2_{k+1,\,k} - h^2_{k,\,k-1}) \frac{h_j}{2N}$$
$$- (h_{k+1,\,k} - h_{k,\,k-1})(h^2_{j+1,\,j} - h^2_{j,\,j-1}) \frac{h_k}{2N} \Big] \Big\},$$

where n_j includes n_k. Hence

$$r^2_{x.c_x} \sigma^2_{r_{x.c_x}} = \sum_{s=1}^{p-1} S \Big\{ \frac{n_s}{N^2}\Big(1 - \frac{n_s}{N}\Big)(h_{s+1,\,s} - h_{s,\,s-1})^2 [h_s - \tfrac{1}{2}(h_{s+1,\,s} + h_{s,\,s-1})]^2 \Big\}$$
$$+ \sum_{j,k}{}' S \Big\{ \frac{2n_k}{N^2}\Big(1 - \frac{n_j}{N}\Big)(h_{j+1,\,j} - h_{j,\,j-1})(h_{k+1,\,k} - h_{k,\,k-1}) [h_j - \tfrac{1}{2}(h_{j+1,\,j} + h_{j,\,j-1})] \times [h_k - \tfrac{1}{2}(h_{k+1,\,k} + h_{k,\,k-1})] \Big\} \quad \ldots\ldots\text{(xi)},$$

where the *first summation* is for all values of s from 1 to $p-1$; e.g. if the data are divided into 7 categories giving dichotomies at $-\infty, h_1, h_2 \ldots h_6, +\infty$ there will be six terms in the summation.

The *second summation* is for all possible pairs of integers (j not equal to k) between 1 and $p-1$; there will thus be $\frac{(p-1)(p-2)}{2}$ terms in the summation, e.g. if $p=7$, there will be 15 terms.

In order to calculate the standard deviation and therefore (on the assumption of the normality of the distribution of $\delta r_{x.c_x}$) the probable error of a class-index correction, the following expressions are required for each dichotomy, h_s:

(1) $\frac{n_s}{N}=\alpha_s$, (2) $1-\frac{n_s}{N}=\beta_s$,

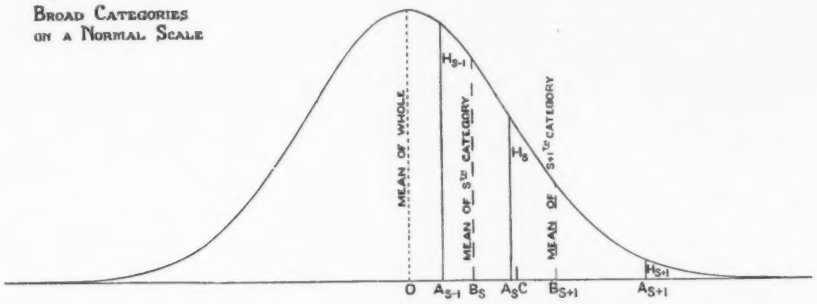
(3) $h_{s+1,\,s}-h_{s,\,s-1}=\gamma_s$, (4) $h_s-\frac{1}{2}(h_{s+1,\,s}+h_{s,\,s-1})=\delta_s$,

whence we have from (xi)

$$\sigma_{r_{c_x}}=\frac{1}{\sqrt{N}}\frac{1}{r_{c_x}}\sqrt{\sum_{s=1}^{p-1}(\alpha_s\beta_s\gamma_s^2\delta_s^2)+2\sum_{j,k}'(\alpha_k\beta_j\gamma_j\gamma_k\delta_j\delta_k)} \quad \text{.....(xii).}$$

In Figure I, $OA_{s-1}=h_{s-1}$, $OA_s=h_s$, $OA_{s+1}=h_{s+1}$,
$OB_s=h_{s,\,s-1}$, $OB_{s+1}=h_{s+1,\,s}$,
C is the mid-point of B_sB_{s+1}.

Fig. I
BROAD CATEGORIES ON A NORMAL SCALE



Then $\gamma_s=B_sB_{s+1}$, $\delta_s=CA_s$. Both γ_s and δ_s will decrease as the number of groups is increased: δ_s is likely to be a very small quantity unless we are dealing with a few unequal groups. These points will be illustrated in the numerical examples given below.

(5) The problem is not completed, until we have considered the manner in which the errors in $r_{x.c_x}$ and $r_{y.c_y}$ affect the corrected coefficients of correlation or of mean square contingency, or the corrected correlation ratio. We know that

$$r_{xy}=\frac{r_{c_x.c_y}}{r_{x.c_x}r_{y.c_y}} \quad \text{.....(xiii),}$$

where $r_{c_x.c_y}$ is the coefficient of correlation between the categories or class-marks of

the variates, and r_{xy} is the correlation between their true quantitative values. Or again using the coefficient mean square contingency

$$r_{xy} = \frac{C_2}{r_{x.c_x} r_{y.c_y}} \quad \text{.....(xiv).}$$

Lastly if we are calculating a correlation ratio, and H_{yc_x} is the value found when the y variate is finely classified so that the mean of an array can be determined, while the x variate is divided into broad classes only, then the corrected η is given by

$$\eta_{yx} = \frac{H_{yc_x}}{r_{x.c_x}} \quad \text{.....(xv).}$$

If we take logarithmic differentials* of (xiv) and square both sides of the equation, we have

$$\frac{\delta r^2_{xy}}{\tilde{r}^2_{xy}} = \frac{\delta C_2^2}{\tilde{C}_2^2} + \frac{\delta r^2_{x.c_x}}{\tilde{r}^2_{x.c_x}} + \frac{\delta r^2_{y.c_y}}{\tilde{r}^2_{y.c_y}} - \frac{2\delta C_2 \delta r_{x.c_x}}{\tilde{C}_2 \tilde{r}_{x.c_x}} - \frac{2\delta C_2 \delta r_{y.c_y}}{\tilde{C}_2 \tilde{r}_{y.c_y}} + \frac{2\delta r_{x.c_x} \delta r_{y.c_y}}{\tilde{r}_{x.c_x} \tilde{r}_{y.c_y}},$$

where $\tilde{r}_{xy}$, $\tilde{C}_2$, $\tilde{r}_{x.c_x}$ and $\tilde{r}_{y.c_y}$ are the values of the constants in the sampled population, and the δ's denote the variation from these values of the constants calculated from the material of a sample. If we sum for all possible variations, and take the mean value of both sides of the equation, we have

$$\sigma^2_{r_{xy}} = \tilde{r}^2_{xy} \left\{ \frac{\sigma^2_{c_2}}{\tilde{C}_2^2} + \frac{\sigma^2_{r_{x.c_x}}}{\tilde{r}^2_{x.c_x}} + \frac{\sigma^2_{r_{y.c_y}}}{\tilde{r}^2_{y.c_y}} - \frac{2 \text{ Mean } (\delta C_2 \, \delta r_{x.c_x})}{\tilde{C}_2 \tilde{r}_{x.c_x}} \right.$$
$$\left. - \frac{2 \text{ Mean } (\delta C_2 \, \delta r_{y.c_y})}{\tilde{C}_2 \tilde{r}_{y.c_y}} + \frac{2 \text{ Mean } (\delta r_{x.c_x} \delta r_{y.c_y})}{\tilde{r}_{x.c_x} \tilde{r}_{y.c_y}} \right\} \quad \text{.....(xvi).}$$

The calculation of the three mean value expressions is likely to be complex, and it is not proposed to make the attempt in the present paper. The correlations between the values of $\tilde{C}_2$, $\tilde{r}_{x.c_x}$ and $\tilde{r}_{y.c_y}$ must however lie between + 1 and − 1, so that it follows that

$$\sigma_{r_{xy}} \leqslant \tilde{r}_{xy} \left\{ \frac{\sigma_{c_2}}{\tilde{C}_2} + \frac{\sigma_{r_{x.c_x}}}{\tilde{r}_{x.c_x}} + \frac{\sigma_{r_{y.c_y}}}{\tilde{r}_{y.c_y}} \right\} \quad \text{.....(xvii).}$$

In the same way it follows from (xv) that

$$\sigma_{\eta_{y_x}} \leqslant \tilde{\eta}_{y_x} \left\{ \frac{\sigma_{H_{yc_x}}}{\tilde{H}_{yc_x}} + \frac{\sigma_{r_{x.c_x}}}{\tilde{r}_{x.c_x}} \right\} \quad \text{.....(xviii).}$$

As the values of the constants in the sampled population will generally be unknown, we shall as usual in (xvii) and (xviii) put for $\tilde{C}_2$, $\tilde{r}_{x.c_x}$, $\tilde{r}_{y.c_y}$, etc. the values of these constants found in the sample. The results of the examples worked out below suggest that the standard deviations of the class-index corrections $r_{x.c_x}$ and

* As before we assume that statistical differentials may be treated as mathematical differentials, or that $\frac{1}{\sqrt{N}}$ is small compared with unity.

$r_{y.c_y}$ are in general likely to be so small compared to the standard deviations of H_{yc_x} and C_2, that rough approximations for the upper limits of the probable errors sufficient for practical purposes will be:

(*a*) for r_{xy} found by the corrected mean square contingency method,

$$\pm \cdot 67449 \frac{r_{xy}}{C_2} \sigma_{c_2} = \pm \cdot 67449 \frac{\sigma_{c_2}}{r_{x.c_x} r_{y.c_y}},$$

(*b*) for η_{yx}, $\pm \cdot 67449 \frac{\eta_{yx}}{H_{yc_x}} \sigma_{H_{yc_x}} = \pm \cdot 67449 \frac{\sigma_{H_{yc_x}}}{r_{x.c_x}}$.

(6) *Illustrative Examples.*

Example 1.

At the top of the table below is given a distribution, divided into five broad classes, of the health among 1918 schoolboys*.

TABLE I.

Health in Schoolboys (*5 categories*).

Very Strong	Strong	Normally Healthy	Rather Delicate	Delicate	Total
70·5	605·5	907	313	22	1918

s	1	2	3	4
$\alpha_s = \frac{n_s}{N}$	·96324	·64755	·17466	·01147
$\beta_s = 1 - \frac{n_s}{N}$	·03676	·35245	·82534	·98853
h_s	−1·7896	−·3787	+·9359	+2·2744

r, s	1, 0	2, 1	3, 2	4, 3	5, 4
$h_{r,s}$	−2·1882	−·9214	+·2408	+1·3937	+2·6181

s	1	2	3	4
γ_s	+1·2668	+1·1622	+1·1529	+1·2244
δ_s	−·2348	−·0384	+·1187	+·2685

* Taken from the table in *Biometrika*, Vol. III. p. 166.

If these data are fitted to a normal scale, we find the values given in the table for the proportional frequencies $\frac{n_s}{N}$ and $1-\frac{n_s}{N}$, and for the positions of the dichotomic lines, h_s, and of the centroids of the groups, $h_{r,s}$. Putting these values into Equation (x) and remembering that $1-\frac{n_0}{N}$ and $\frac{n_5}{N}$ are zero, it is found that the probable errors of the distances of the centroids or means of the groups from the mean of the sample (referred to the standard deviation as unit) are for h_{10}, $\pm$ ·0314; for h_{21}, $\pm$ ·0166; for h_{32}, $\pm$·0149; for h_{43}, $\pm$ ·0204 and for h_{54}, $\pm$ ·0492. The largest probable errors occur in the small groups at the tails of the distribution.

The relation
$$r^2_{x.c_x} = \sum_{s=0}^{4} \left\{ \frac{n_s - n_{s+1}}{N} h^2_{s+1,s} \right\}$$
gives
$$r^2_{x.c_x} = \cdot 867055, \qquad r_{x.c_x} = \cdot 9312.$$

To find the probable error of $r_{x.c_x}$ we calculate first the values of the auxiliary qtuantities γ_s and δ_s (defined on p. 266); these are given in the last two rows of Table I; α_s and β_s or $\frac{n_s}{N}$ and $1-\frac{n_s}{N}$ have been calculated already. Substituting these expressions in Equation (xii) it is found first that

$$\sum_{s=1}^{4} (\sigma_s \beta_s \gamma_s{}^2 \delta_s{}^2) = + \cdot 007513, \quad \sum'_{j,k} (\alpha_k \beta_j \gamma_j \gamma_k \delta_j \delta_k) = + \cdot 000005,$$

and finally,
$$\sigma_{r_{x.c_x}} = \cdot 00213.$$

Hence on the assumption that the distribution of variations in samples of $r_{x.c_x}$ may be treated as approximately normal, the probable error of $r_{x.c_x}$ is $\pm$ ·0014, or
$$r_{x.c_x} = \cdot 9312 \pm \cdot 0014.$$

Example 2.

In order to see how a reduction in the number of groups affects the probable errors, I have combined the Very Strong and Strong groups, and the Rather Delicate and Delicate groups. The altered values of the constants are given in Table II. The probable errors of the $h_{r,s}$'s are now

for h_{10}, $\pm$ ·0141; for h_{21}, $\pm$ ·0149; for h_{32}, $\pm$ ·0180,

and the class-index correction is $r_{x.c_x} = \cdot 8934$,

$$\sum_{s=1}^{3} (\alpha_s \beta_s \gamma_s{}^2 \delta_s{}^2) = + \cdot 001644, \quad \sum'_{j,k} (\alpha_k \beta_j \gamma_j \gamma_k \delta_j \delta_k) = \alpha_2 \beta_1 \gamma_1 \gamma_2 \delta_1 \delta_2 = + \cdot 000427,$$

whence $\sigma_{r_{x.c_x}} = \cdot 00116$, giving a probable error of $\pm$ ·0008.

If these results are compared with those of Example 1, it will be seen that by getting rid of the small tail groups, the probable errors of the means have been reduced and at the same time, while the class-index correction itself has a lower value, its probable error is considerably smaller in the second system of grouping. The low value for $\sigma_{r_{x.c_x}}$ is remarkable; its value has been checked by another method described in section (7) below. Now suppose that we had some other character of the 1918 schoolboys—weight, for example—and that we had found a

TABLE II.

Health in Schoolboys (3 *categories*).

	Very Strong and Strong	Normally Healthy	Rather Delicate and Delicate	Total
	676	907	335	1918

s	1	2
$\alpha_s = \frac{n_s}{N}$	·64755	·17466
$\beta_s = 1 - \frac{n_s}{N}$	·35245	·82534
h_s	− ·3787	+ ·9359

r, s	1, 0	2, 1	3, 2
$h_{r,s}$	−1·0536	+ ·2408	+1·4741

s	1	2
γ_s	+1·2944	+1·2333
δ_s	+ ·0277	+ ·0785

correlation ratio, H_{yc_x}, of ·40, between this character and the five health class-marks; the probable error of H_{yc_x} would be given approximately as

$$\pm \cdot 67449 \frac{1 - H^2}{\sqrt{N}} = \pm \cdot 0129,$$

while the corrected $\eta_{yx} = \frac{\cdot 4000}{\cdot 9312} = \cdot 4296$. Then (xviii) gives the upper limit of the probable error of η_{yx} as

$$\pm \frac{1}{r_{x.c_x}} (\text{P.E. of } H_{yc_x} + \eta_{yx} \times \text{P.E. of } r_{x.c_x})$$

$$= \pm \frac{1}{\cdot 9312} (\cdot 0129 + \cdot 4296 \times \cdot 0014) = \pm \cdot 0145.$$

Thus the probable error is for rough purposes given by

$$\pm \cdot 67449 \frac{\sigma_{H_{yc_x}}}{r_{x.c_x}} = \pm \cdot 0139.$$

Example 3.

In this example I have taken data of non-Gaussian frequency, divided into groups of very unequal distribution. These are the figures for barometric heights at Southampton and Laudale distributed in a 3 × 3-fold table, as below*:

TABLE III. *Frequency of Barometric Heights (in inches).*

Southampton.

Laudale.	*31·05—30·15*	*30·15—29·25*	*29·25—28·35*	Totals
30·85—29·85	808·25	733·25	0	1541·5
29·85—28·85	83·75	1223·25	45·5	1352·5
28·85—27·85	0	14	14	28·0
Totals	892·0	1970·5	59·5	2922

The Southampton distribution will correspond to x, and the Laudale to y. Then we find for the Southampton marginal totals:

TABLE IV.

Southampton.

s	1	2
$\alpha_s=\frac{n_s}{N}$	·69473	·02036
$\beta_s=1-\frac{n_s}{N}$	·30527	·97964
h_s	− ·5093	+2·0463
γ_s	+1·5946	+1·9673
δ_s	− ·1587	+ ·6160

r, s	1, 0	2, 1	3, 2
$h_{r,s}$	−1·1479	+·4467	+2·4140

Giving $r_{x.c_x}=$ ·8096 and $\sigma_{r_{x.c_x}}=$ ·00452,

and the probable error ± ·0030.

* The table is taken from *Biometrika*, Vol. IX. p. 137 from the paper on the Influence of "Broad Categories" on Correlation. Slips which there seem to have occurred in the calculation of $r_{x.c_x}$ and $r_{y.c_y}$ have now been corrected. The values of $h_{r,s}$ and $k_{r,s}$ do not quite agree to the last figure with those given in the paper quoted, but the present values were obtained using the "Tables of Deviates and of Ordinates of the Normal Curve for each Permille of Frequency," *Tables for Statisticians and Biometricians*, Cambridge University Press, Table I, and *Biometrika*, Vol. XIII. p. 428 respectively, while the original values were probably obtained by working to more figures. The difference is not significant, as it makes no difference to the value of $r_{x.c_x}$ or $r_{y.c_y}$ calculated to four figures.

For Laudale, we have Table V:

TABLE V.

Laudale.

s	1	2
$\alpha_s' = \frac{n_s}{N}$	·47245	·00958
$\beta_s' = 1 - \frac{n_s}{N}$	·52755	·99042
k_s	+ ·0691	+2·3424
γ_s'	+1·5585	+1·8753
δ_s'	+ ·0441	+ ·6006

r, s	1, 0	2, 1	3, 2
$k_{r,s}$	−·7543	+·8042	+2·6795

Giving $r_{y.c_y} = ·8175, \quad \sigma_{r_{y.c_y}} = ·00268,$

and the probable error $\pm ·0018.$

A glance at the marginal totals would suggest that the y-distribution with its small tail group of 28 would give a less reliable class-index correction than the x-distribution. That $\sigma_{r_{y.c_y}}$ is actually smaller than $\sigma_{r_{x.c_x}}$ is mainly due to the fact that $\delta_1'(+·0441)$ is less than one-third of $\delta_1(-·1587)$, the mean of k_{10} and k_{21} nearly coinciding in position with k_1. This is illustrated diagrammatically in Figure II.

Now using the method of mean square contingency I find that $\phi^2 = ·3662\ 9808$ and $C_2 = ·51778$, hence

$$r_{xy} = \frac{C_2}{r_{x.c_x} r_{y.c_y}} = ·7823.$$

In order to find the probable error of C_2, I have calculated σ_{ϕ^2} by the method of Pearson and Young, *Biometrika*, Vol. XI. pp. 215—230*, and obtain

$$\sigma_{\phi^2} = ·064745,$$

whence $$\sigma_\phi = \frac{1}{2\phi}\sigma_{\phi^2} = ·053488,$$

and $$\sigma_{c_2} = \sigma_\phi/(1+\phi^2)^{\frac{3}{2}} = ·033492,$$

giving finally the probable error of C_2 as $\pm ·02259$.

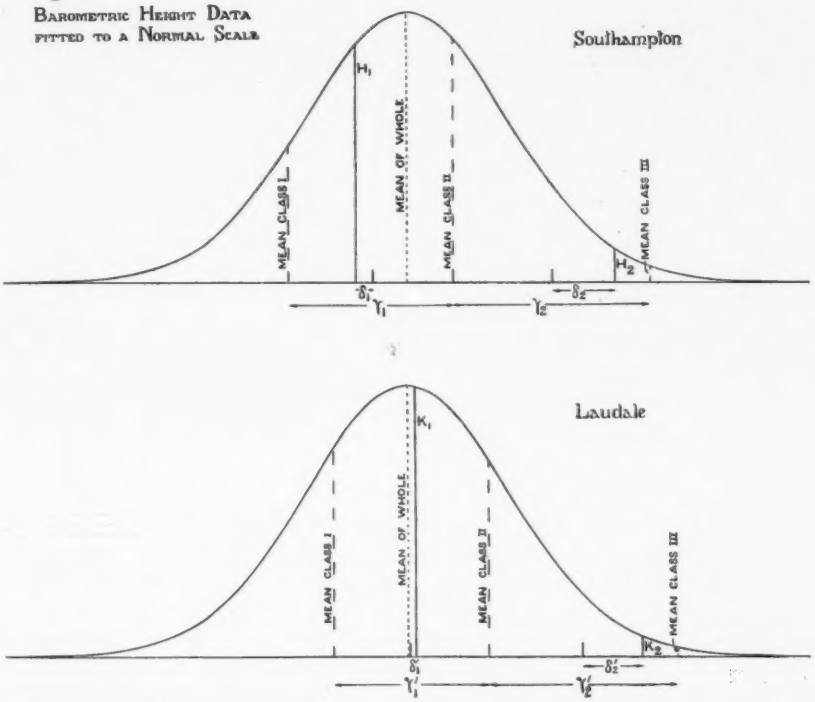
* I have used the approximative form (D) of p. 229 of that paper, where the sign of $2c$ in the second bracket should be negative, as corrected in *Biometrika*, Vol. XII. p. 259.

If now these values of C_2, $r_{x.c_x}$ and $r_{y.c_y}$ and of their probable errors are put into the relation (xvii), we have for the upper limit of the probable error of the corrected r_{xy}*,

$$\pm \cdot 7823 \left(\frac{\cdot 0226}{\cdot 5178} + \frac{\cdot 0030}{\cdot 8096} + \frac{\cdot 0018}{\cdot 8175}\right)$$
$$= \pm (\cdot 0341 + \cdot 0029 + \cdot 0017)$$
$$= \pm \cdot 0387.$$

Fig. II

BAROMETRIC HEIGHT DATA FITTED TO A NORMAL SCALE



Here again a rough approximation to the probable error is given by the first term, or

$$\pm \cdot 67449 \frac{\sigma_{c_2}}{r_{x.c_x} r_{y.c_y}}.$$

Example 4.

In the previous examples the total number N in the sample has been fairly large. To examine the value of the probable error of $r_{x.c_x}$ in a smaller sample, and

* Making the correction to the value of r_{xy} given in the previous paper referred to above (*Biometrika*, Vol. IX. pp. 137, 138), we have to compare the value found by corrected mean square contingency, (·7823) instead of (·7570), with that found by the product moment method (·7802 ± ·0049).

its relation to the probable error of C_2, I have taken the contingency table originally given by Miss Elderton in her Memoir on *The Measure of Resemblance of First Cousins* *, for which C_2 and its probable error have been calculated by Young and Pearson†. This is a 6×6-fold table, but as it is symmetrical about a diagonal, $r_{x.c_x}$ and $r_{y.c_y}$ are the same.

TABLE VI.

Contingency between Hair Colours of Female Cousins.

First Female Cousin.

Second Female Cousin. Tints	Very Dark	Dark Brown	Brown	Light Brown	Fair	Red	Totals
Very Dark...	9	11	3·5	11	1·5	0	36
Dark Brown	11	13	12	10	9	1	56
Brown ...	3·5	12	8	9·75	3·25	3	39·5
Light Brown	11	10	9·75	16·5	9·25	1	57·5
Fair... ...	1·5	9	3·25	9·25	1	0	24
Red	0	1	3	1	0	0	5
Totals	36	56	39·5	57·5	24	5	218

The values of the constants h_s, $h_{r,s}$ and α_s, β_s, γ_s, δ_s are given in Table VII. From these it is found that

$$r_{x.c_x} = \cdot 9626, \quad \sum_{s=1}^{5} (\alpha_s \beta_s \gamma_s^2 \delta_s^2) = \cdot 0017036, \quad \sum'_{j,k} (\alpha_k \beta_j \gamma_j \gamma_k \delta_j \delta_k) = -\cdot 0002368,$$

TABLE VII.

s	1	2	3	4	5
$\alpha_s = \frac{n_s}{N}$	·83486	·57798	·39679	·13303	·02294
$\beta_s = 1 - \frac{n_s}{N}$	·16514	·42202	·60321	·86697	·97706
h_s	−·9735	−·1967	+·2616	+1·1122	+1·9965
γ_s	+·9476	+·5882	+·6150	+·8116	+·9117
δ_s	+·0567	+·0656	−·0777	+·0596	+·0823

r, s	1, 0	2, 1	3, 2	4, 3	5, 4	6, 5
$h_{r,s}$	−1·5040	−·5564	+·0318	+·6468	+1·4584	+2·3701

* *Eugenics Laboratory Memoirs*, IV. Cambridge University Press, 1907. The group of 5 with red hair is undesirably small.

† *Biometrika*, Vol. XI. pp. 225—227 and correction for error in *Biometrika*, Vol. XII. p. 260.

and hence $\sigma_{r_{x.c_x}} = \cdot00247$, and the probable error of $r_{x.c_x}$ is $\pm\ \cdot0017$, or is only slightly greater than in Example 1, where N was 1918 instead of the present 218.

Now $\phi^2 = \cdot14895$, $C_2 = \cdot36005$, $r_{xy} = \dfrac{C_2}{r_{x.c_x} r_{y.c_y}} = \cdot3886$, $\sigma_{\phi^2} = \cdot0719$, $\sigma_{c_2} = \cdot0756$, and the probable error of $C_2 = \pm\ \cdot0510$.

Using relation (xvii), we find the upper limit of the probable error of the corrected r_{xy} to be

$$\pm\ \cdot3886 \left(\frac{\cdot0510}{\cdot3600} + 2 \times \frac{\cdot0017}{\cdot9626}\right)$$

$$= \pm\ (\cdot0550 + \cdot0014) = \pm\ \cdot0564,$$

where the second term due to the probable errors of the class-index correction is again insignificant compared to the first, due to the probable error of C_2.

(7) In section (4), $\delta r_{x.c_x}$, or a variation in the class-index correction found from a sample out of the population, was expressed in terms of the variations in H_s and n_s; it can also be expressed in terms of the variations in $h_{s+1,s}$ and n_s. Thus differentiating (iii),

$$2r_{x.c_x}\delta r_{x.c_x} = \sum_{s=0}^{s=p-1} \left\{ \frac{\delta n_s - \delta n_{s+1}}{N} h^2_{s+1,s} + 2h_{s+1,s}\,\delta h_{s+1,s} \frac{n_s - n_{s+1}}{N} \right\} \ldots\ldots \text{(xix)}.$$

If we square both sides of (xix), sum for all possible variations in samples, and take the mean value, we shall be able to express $\sigma^2_{r_{x.c_x}}$ in terms of the mean values of the squares and products of the differentials of n_s and $h_{s+1,s}$, $h_{r+1,r}$, etc., or of

$$\text{Mean}\,(\delta n_s^2) = \sigma^2_{n_s}, \quad \text{Mean}\,(\delta n_s \delta n_r), \quad \text{Mean}\,(\delta h^2_{s+1,s}) = \sigma^2_{h_{s+1,s}},$$

$$\text{Mean}\,(\delta n_s\,\delta h_{s+1,s}), \quad \text{Mean}\,(\delta n_s\,\delta h_{r+1,r}), \text{ etc.}$$

Of these the first three expressions are known from Equations (i), (ii) and (x), and the others can be calculated in terms of n_s, h_s, $h_{s+1,s}$, etc. As a check the standard deviation of $r_{x.c_x}$ was worked out by this method for the simple case of the three groupings of Example 2 above. It was found to be $\cdot00116$, agreeing exactly with the value given on p. 269. This appears to give a satisfactory confirmation of the correctness of both Equations (x) and (xi) as well as of the accuracy of the arithmetic in this particular example.

(8) There is another method of calculating the probable error of $r_{x.c_x}$ if the data are recorded on a *quantitative* scale. In this case $r_{x.c_x} = \dfrac{\sigma_{c_x}}{\sigma_x}$, where σ_{c_x}, the standard deviation of the means of the groups, and σ_x, the standard deviation of the whole sample, are both known numerically. The problem is broadly the same as that of finding the probable error of a correlation ratio, η, for both $r_{x.c_x}$ and η are the ratios of the standard deviation of the means of independent groups to the standard deviation of the total distribution. Now the standard deviation of the values of η due to random sampling has been worked out in a previous paper*, and

* *Drapers' Company Research Memoirs.* No. XIV. "On the General Theory of Skew Correlation and Non-linear Regression." 1905.

with slight modification the relations there obtained will give the standard deviation of $r_{x.c_x}$.

We have first that

$$\frac{\delta r_{x.c_x}}{r_{x.c_x}} = \frac{\delta\sigma_{c_x}}{\sigma_{c_x}} - \frac{\delta\sigma_x}{\sigma_x},$$

and on squaring, summing for all random samples and dividing by the number of such samples, we find:

$$\frac{\sigma^2_{r_{x.c_x}}}{r^2_{x.c_x}} = \frac{\sigma^2_{\sigma_{c_x}}}{\sigma^2_{c_x}} + \frac{\sigma^2_{\sigma_x}}{\sigma^2_x} - \frac{2\,\text{Mean}\,(\delta\sigma_{c_x}\,\delta\sigma_x)}{\sigma_{c_x}\,\sigma_x} \quad \text{.....(xx).}$$

Now we know that*

$$\sigma^2_{\sigma_x} = \frac{\sigma_x^2}{4N}\frac{\mu_4 - \mu_2^2}{\mu_2^2} = \frac{\sigma_x^2}{4N}(\beta_2 - 1) \quad \text{.....(xxi),}$$

where μ_4 and μ_2 are the 4th and 2nd moment coefficients of the sample.

Making the necessary modification to Equation (xxvii) of p. 16 of the Memoir referred to in the footnote on p. 275, we find that

$$\frac{\sigma^2_{\sigma_{c_x}}}{\sigma^2_{c_x}} = \frac{1}{4N}\left\{\frac{\lambda_4 - \lambda_2^2}{\lambda_2^2} + 4\,S_s\left(\frac{m_s(\bar{x}_s - \bar{x})^2\,{}_s\pi_2}{N\lambda_2^2}\right)\right\} \quad \text{.....(xxii),}$$

where $\bar{x}$ is the mean of the sample,

$\bar{x}_s$ is the mean of the sth class or category into which the material is grouped,

m_s is the size of the sth category,

${}_s\pi_q = \dfrac{1}{m_s}S_t\{n_{st}(x_{st} - \bar{x}_s)^q\}$ or is the qth moment coefficient of the observations in the sth category about the mean of that category,

$\lambda_q = \dfrac{1}{N}S_s\{m_s(\bar{x}_s - \bar{x})^q\}$ or is the qth moment coefficient of the weighted means of categories about the mean of the whole sample,

S_s indicates summation for all categories,

S_t indicates summation for all variates within the sth category.

Similarly after modification, Equation (xxxii) of the Memoir gives

$$\frac{\text{Mean}\,(\delta\sigma_{c_x}\,\delta\sigma_x)}{\sigma_{c_x}\,\sigma_x} = \frac{1}{4N}\left[\frac{\lambda_4}{\sigma_x^2\lambda_2} - 1 + 5\,S_s\left\{\frac{m_s(\bar{x}_s - \bar{x})^2\,{}_s\pi_2}{N\sigma_x^2\lambda_2}\right\} + 2\,S_s\left\{\frac{m_s(\bar{x}_s - \bar{x})\,{}_s\pi_3}{N\sigma_x^2\lambda_2}\right\}\right]$$
$$\text{.....(xxiii).}$$

Substituting from (xxi), (xxii) and (xxiii) into (xx), we find that

$$\sigma^2_{r_{x.c_x}} = \frac{r^2_{x.c_x}}{4N}\left[\beta_2 + \frac{\lambda_4}{\lambda_2^2}\left(1 - 2\frac{\lambda_2}{\sigma_x^2}\right) + 2\left(2 - 5\frac{\lambda_2}{\sigma_x^2}\right)S_s\left\{\frac{m_s(\bar{x}_s - \bar{x})^2\,{}_s\pi_2}{N\lambda_2^2}\right\}\right.$$
$$\left. - 4\,S_s\left\{\frac{m_s(\bar{x}_s - \bar{x})\,{}_s\pi_3}{N\sigma_x^2\lambda_2}\right\}\right],$$

* *Biometrika*, Vol. II. p. 276.

TABLE VIII. *Barometric Heights.*

	Southampton																	
Height	31·0	30·9	30·8	30·7	30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9	29·8	29·7	29·6	29·5	29·4	29·3
Frequency	1	4	4	30·5	52·5	107·5	140·5	237	315	395·5	382·5	339·5	288	201	150·5	98·5	65	50
Group Frequencies of Example 3	892·0									1970·5								

	Southampton								Laudale									
Height	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	30·8	30·7	30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9
Frequency	23·5	15·5	7·5	7	3	—	1	2	2	14	36	64	141	200	263	260·5	277·5	283·5
Group Frequencies of Example 3	59·5								1541·5									

	Laudale																		
Height	29·8	29·7	29·6	29·5	29·4	29·3	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	28·4	28·3	28·2	28·1	28·0	27·9
Frequency	277·5	245	212	192	135	97·5	67·5	63	38·5	24·5	11	7·5	4·5	—	2·5	0·5	—	—	1	1
Group Frequencies of Example 3	1352·5										28·0									

Total for both Southampton and Laudale = 2922.

or, remembering that

$$r^2_{x.c_x} = \frac{\sigma^2_{c_x}}{\sigma_x^2} = \frac{\lambda_2}{\sigma_x^2},$$

and putting

$$\frac{\lambda_4}{\lambda_2^2} = B_2,$$

we have finally

$$\sigma^2_{r_{x.c_x}} = \frac{r^2_{x.c_x}}{4N}\left[\beta_2 + B_2(1-2r^2_{x.c_x}) + 2(2-5r^2_{x.c_x})\underset{s}{S}\left\{\frac{m_s(\bar{x}_s-\bar{x})^2\,{}_s\pi_2}{N\sigma^4_{c_x}}\right\} - 4\underset{s}{S}\left\{\frac{m_s(\bar{x}_s-\bar{x})\,{}_s\pi_3}{N\sigma_x^2\,\sigma^2_{c_x}}\right\}\right] \quad \text{.....(xxiv).}$$

In this equation we cannot, as in the case of the standard deviation of η, make any assumption as to the values of B_2, ${}_s\pi_2$ or ${}_s\pi_3$. When dealing with a few categories only, ${}_s\pi_3$ is likely to be quite significant particularly in the unsymmetrical tail groups.

(9) As an illustration of Equation (xxiv) we may take the barometric-height data of Example 3, above. All that is required for the present purpose are the marginal frequencies for Southampton and Laudale*; these are given in Table VIII, and in Table IX are given the constants required in (xxiv) calculated from these frequencies. Substituting in (xxiv) it is found that,

for Southampton $\sigma_{r_{x.c_x}} = \cdot00608$,

for Laudale $\sigma_{r_{y.c_y}} = \cdot00386$.

TABLE IX.

	Southampton			Laudale		
	$\bar{x}=29''\cdot9839$; $\sigma_x^2=\cdot105629$; $\beta_2=3\cdot612029$			$\bar{x}=29''\cdot8488$; $\sigma_x^2=\cdot154629$; $\beta_2=3\cdot194947$		
	Group 1	Group 2	Group 3	Group 1	Group 2	Group 3
$\bar{x}_s$	$30''\cdot3416$	$29''\cdot8499$	$29''\cdot0597$	$30''\cdot1503$	$29''\cdot5304$	$28''\cdot6357$
${}_s\pi_2$	$\cdot022250$	$\cdot045256$	$\cdot029129$	$\cdot038140$	$\cdot055822$	$\cdot052653$
${}_s\pi_3$	$+\cdot003611$	$-\cdot007100$	$-\cdot007808$	$+\cdot004128$	$-\cdot010449$	$-\cdot023435$
	$\sigma^2_{c_x}=\cdot068561$; $r^2_{x.c_x}=\cdot649070$; $B_2=4\cdot2699$			$\sigma^2_{c_y}=\cdot108982$; $r^2_{x.c_y}=\cdot704797$; $B_2=2\cdot5148$		

If these values are compared with those found above from (xi), by the method of fitting the broad group totals to a normal scale, viz.

$$\sigma_{r_{x.c_x}} = \cdot00452,$$

$$\sigma_{r_{y.c_y}} = \cdot00268,$$

* The complete data are given in Table IX, p. 453 of the paper by Pearson and Lee in *Phil. Trans.* Vol. 190, A, 1898.

it will be seen that in both cases the standard deviation found from (xxiv) is about 1·4 times as great as that found from (xi).

It appears therefore that by fitting our material to a normal scale and not dealing with the crude data, we have ensured a slightly greater degree of consistency in the values of the class-index correction calculated from different samples of N. If we compare the values of $r_{x.c_x}$ found by the two methods we find

	Southampton	Laudale
Qualitative method	·8096	·8175
Quantitative method	·8056	·8395

There is a significant difference between the two values of the class-index correction for the Laudale distribution, but if the very small size of the tail group (i.e. 28·0) is remembered this discrepancy is perhaps not surprising.

It is also possible to use Equation (xxiv) when dealing with material which is divided into qualitative categories only. For in this case, fitting the groups as before to a normal scale, we have

$$\beta_2 = 3\,; \quad B_2 = \frac{1}{r^4_{x.c_x}} S_s \left\{ \frac{(n_s - n_{s+1})}{N} h^4_{s+1,s} \right\};$$

$$S_s \left\{ \frac{m_s (\bar{x}_s - \bar{x})^2 {}_s\pi_2}{N \sigma^4_{c_x}} \right\} = \frac{1}{r^4_{x.c_x}} S_s \left\{ h^2_{s+1,s} (n_s - n_{s+1}) {}_s\pi_2' \right\};$$

$$S_s \left\{ \frac{m_s (\bar{x}_s - \bar{x}) {}_s\pi_3}{N \sigma_x{}^2 \sigma^2_{c_x}} \right\} = \frac{1}{r^2_{x.c_x}} S_s \left\{ h_{s+1,s} (n_s - n_{s+1}) {}_s\pi_3' \right\},$$

where ${}_s\pi_q'$ is the qth moment coefficient about its mean ordinate of the section of the normal curve

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2},$$

lying between the dichotomic ordinates at h_s and h_{s+1}.

Taking the Southampton distribution, and making use of the values of $h_{s+1,s}$, h_s, $r_{x.c_x}$, etc., found in the working of Example 3, I calculated ${}_s\pi_2'$ and ${}_s\pi_3'$ for each of the three categories by a rough quadrature. Equation (xxiv) then gave

$$\sigma_{r_{x.c_x}} = \cdot 0049.$$

We may thus compare the values of the standard deviation of $r_{x.c_x}$ for the Southampton distribution found by the three methods:

(1) from (xi) using broad categories only and fitting to a normal scale ·0045,
(2) „ (xxiv) „ „ „ „ ·0049,
(3) „ (xxiv) using the crude quantitative data ... ... ·0061.

As should be the case the results of methods (1) and (2) agree closely. Without further numerical testing it cannot be asserted that for all other arrangements of grouping the class-index correction found by the broad category method will have a smaller probable error than that calculated using the true quantitative scale, but since the arrangement of grouping could hardly be more unfavourable than in the

present example it seems probable that the results of these two lines of approach will not in general differ seriously. Method (1) is undoubtedly the shortest, and in the majority of problems in which the class-index correction is required, the quantitative scaling of the variate required in method (3) will not be available.

(10) *Conclusion.* Expressions have been found for the probable error due to random sampling of the distance (in terms of the standard deviation as unit) between the mean of a class group or broad category and the mean of the whole sample, in material fitted to a normal scale, (x); and for the probable error of a class-index correction, (xi), (xii), and (xxiv). The first problem is straightforward and calls for no comment. The second has been approached from several points of view, and illustrated by examples covering a fairly wide range, which suggest that the probable errors of $r_{x.c_x}$ and $r_{y.c_y}$ are likely to be so small compared to the probable error of the coefficient to be corrected—whether $r_{c_x.c_y}$, H_{yc_x} or C_z—that they may be neglected in the rough appreciation of the probable error of the corrected constant which is all that is usually required for practical purposes. The greatest values are likely to occur when there are a few unevenly divided groups, but Example 3 with its contingency table (Table III) of exceptionally unfavourable grouping, shows that even in such cases the probable error of the class-index corrections will remain very small provided that the size of the sample is reasonably large.

Thus this paper indicates that a class-index correlation based on broad categories is not subject to a large probable error, but only to an error of the same, or even a less order, than a product moment coefficient of correlation.

ON THE VARIATE DIFFERENCE METHOD.

Being a Paper read before the Society of Biometricians and Statisticians.

By KARL PEARSON, F.R.S. and ETHEL M. ELDERTON, *Galton Fellow.*

(1) Every one is familiar with data which show a secular trend with time, say the deaths from tuberculosis as measured by the corrected deathrate from 1865 down to the present or again the falling birthrate per married woman between 15 and 50.

Now the data for such matters do not when plotted take the form of smooth curves corresponding to continuous mathematical functions; they exhibit general trends with the time, but they rise and fall with apparent, but not indeed necessary irregularity, above and below a sort of average curve representing the secular trend. The words "sort of average curve" are vague and intended to be so, because much of the matter in dispute turns on the manner in which the curve representing the secular trend is to be determined. The curve which represents this secular trend may be a long period periodic term or it may not. Generally we have absolutely no reason to suppose it is so, but we may imagine it so, if we have a monomania for the representation of time-functions by periodic analysis*. We have, however, no reason for supposing that increased consumption of apples or bananas, or increasing expenditure on the navy, per head of the population are periodic in their nature; they may rise or fall according to circumstance; but we are no more justified in assuming that they follow a time-function $A \sin(nt + \alpha)$ than $A \sin(nf(t) + \alpha)$, indeed the latter, as it is more general, is more likely to describe the result. A still more general attitude is simply to assume that the trend is given by $f(t)$ itself for this involves no suggestion of periods for $t = f^{-1}\left(\frac{2\pi i}{n}\right)$ of the previous formula.

Of course in any case in which it seemed reasonable to use a harmonic curve, one would naturally do so, but in the cases we have chiefly in mind this is certainly not so, and it is far more reasonable to suppose that the secular trend can be given by a single high order parabola, or by the series of such parabolae involved in a good smoothing process, than by any products of periodic analysis.

Now we know that if we correlate the falling phthisis deathrate with the falling birthrate we shall have a correlation of the order 0·9. But no one is likely to believe there is an *organic* relationship between the two of this order,—any more than one

* Any function, e.g. a triangle, can be fairly closely represented by an adequate number of periodic terms for a finite base. But it does not follow that the representation has any organic significance or would be of the slightest value if used for extrapolation, i.e. prediction.

believes that the correlation between the cancer deathrate and the increasing expenditure on apples per head of the population, the value of which is 0·89, is a true organic relationship, i.e. is due to one or more common factors in the two variates. Such high correlations as arise from common growth or decline with time, when interpreted as causal or semicausal relationships, are in our opinion perfectly idle, indeed are only too apt to be mischievous, and we shall reach nothing, or less than nothing—knighthoods,—by the investigation of them.

But when we take the *apparently* random deviations from the secular trend, it does seem a perfectly legitimate problem to ask: Is there any relationship between them?

If the deviations of two variates from their secular trends be X_t and Y_t, we want to discover their correlation r_{xy}. All are agreed, we think, as to the desirability of finding this correlation,—including even Mr Yule, although he apparently confesses that he cannot find any source for such correlation except in common periodic terms*. Now the real problem before us is this: Having by means of a high order parabola or an adequate smooth got rid of the secular trend, will the variate difference method give us r_{xy} or what does it give us?

There are certain considerations which may be referred to first. As a rule our data are given for annual periods, a year is our usual time unit. Now the most important periodic factor is the annual seasonal change. This with the year as unit will not be of any influence. We know very little yet as to periods in meteorological or climatological phenomena. There are those who talk of an eleven-year period in the former, but it is far from certain, and if it existed, it would for our purposes be of no importance, as its effects would be reduced to insignificance in taking a very few differences. Climatic periodicity seems to be a matter of tens of thousands of years, rather than tens of years, if we may judge by geological considerations; and such periodicity for anything we can say to the contrary may be included in our secular trend. Mr Yule apparently holds that periodicities of 1·2 to 6 times the unit of grouping would be fatal to the variate difference method. It certainly seems to us that any one, who suggests in annual tabling periods of 1·2 to 6 years, is called upon to show how such periods arise, what natural phenomena they are peculiar to, and why such natural causes influence the variates in question. If we take the example already referred to, let us say an isosceles triangle on a time base of a year, then it is clear that a three months periodic term would be an important component in its representation by periodic analysis; but it does not follow that such a period would have the least organic relation to the origin of the triangle. Because we can *represent* a fluctuating variable by a series of periodic terms discovered by periodogram analysis, it does not by any means follow of necessity that the constituent factors of the variate had themselves such periods. The periodogram may *suggest* periods, but that is of small value until the natural

* *Journal of the R. Statistical Society*, Vol. xxxiv. p. 502. Of course any occurrence which affects both variates happens in time, but it is not therefore a function of time, still less a periodic function of time, i.e. it cannot be quantitatively predicted from a knowledge of the time of its occurrence.

factors have been discovered which have these periods, and it can be shown also that these natural factors influence the variates in question.

Again we would add that in our opinion the deviations X and Y from the secular trends of the two variates, such as occur in vital and economic statistics, are dependent on factors which are obviously non-periodic in character. They are summed up in sanitation, legislation, new routes and methods of transport, over- or under-production, new methods of agriculture, wars, famines, transfer of population and thousands of other factors which make up civilised human life. We might define them as "historical factors," history takes place in time, but its events are not mathematical functions of the time, still less periodic functions, whatever folk-experience may whisper about history repeating itself.

It is only legitimate to call the effects of these historical factors random fluctuations, if that term is used in a special sense as Mr Yule appears to use it, i.e. for everything which is not due to a periodic variation. Such "random fluctuations" are by no means as Mr Yule would seem to suggest due only to errors of observation or the deviations of random sampling; they are due to non-periodic causes which may affect both the A and B variates or may not. The question is to what extent have A and B common causes behind their fluctuations, apart from growth with time. We think this is a perfectly legitimate question to ask, and that in asking it we are not open to the insinuation, contained in the use of the term "random variations," of asking whether pure chance fluctuations are or are not correlated*.

Let us suppose for a moment that the distribution of X or Y or of both is purely random, then whether they are in excess or defect of the secular trend would be indifferent, and might be ascertained by a tossing experiment. Now if we take a series of tosses there will be more changes from head to tail or tail to head than anything else, i.e. runs of one are most frequent, then come runs of two and so on, long runs are very rare. But to the casual observer of a graph, runs of one suggest

* An example may possibly render the matter clearer. A financier starts a company to grow cotton in a newly opened African district, and the company in chartering ships to fetch the cotton home finds it of value to export hardware for sale in the district. The company may or may not be successful or may change the nature of its exports and imports. But its contributions to cotton imports and hardware exports are not periodic. Many such transactions—and is not all trade ultimately of this character?—would produce correlated fluctuations in special imports and exports which are not due to periodic factors. Or again take another example. Let us take an English and a Zulu baby, and measure their weights monthly from babyhood to manhood. We should obtain fluctuating values which might be plotted, and the ordinates would give a very high correlation. But if we smoothed these growth data and correlated the fluctuations from the smoothed curves we should anticipate very small correlation. The reason for this would be obvious, the environmental conditions are very different. But if we take the weight curves of two English children and consider the corresponding fluctuations in the same way, we should expect correlation, emphasised if the children were of the same social class and still more if of the same family. There would be a legislative time for vaccination, there would be a traditional or racial time for weaning, there would be local differences in climate and in summer and winter, there would be local epidemics, traditional times for departure for school, for university, for professional life; there would be racial times for puberty, marriage, prime, decay. All these things would leave their impressions on the weight curve. But none of these factors are periodic in the growth curve, yet their relationship as indicated by the correlation of the fluctuations would be of very great interest as measuring the difference between what are generally human and what are especially racial factors.

two-year (or other unit) periods, runs of two, especially if they follow each other, four-year periods, and it is not difficult to be impressed with the presence of short periods by a graph, which is of really random distribution. But when we come more carefully to examine such graphs we find that the runs do not always give a chance distribution. If X_t be in considerable excess or considerable defect, X_{t+1} is more likely to be somewhat in excess or defect also. In other graphs, however, we find a compensating influence in successive years; instead of an excess of permanencies there is an excess of changes. In the former cases we anticipate a positive, in the latter cases a negative correlation between X_t and X_{t+1}. This result is inconsistent with a fundamental hypothesis of the Variate Difference Method as originally stated by "Student" and in this matter it seems to us that there is need to extend the theory.

Now the old method we had adopted in the Biometric Laboratory before the publication of "Student's" memoir was to fit high order parabolae to both variates, and then take the differences between the ordinates of these parabolae and the observed data for X and Y. These were then correlated, and the question of whether or not X_t and X_{t+1} were or were not correlated was immaterial. The difficulty of this process was two-fold: (i) The same parabola was used throughout the whole system; this involved not only a very lengthy piece of work by least squares, but it was obvious that in many cases a change of parabola would be advantageous. Accordingly it is better to replace this high order parabola by a series of parabolae such as are provided by Sheppard's* or Rhodes'† system of smoothing. (ii) It was not *a priori* possible to select aptly the order requisite for the parabola, nor having selected it to settle, except from the general appearance of the graph, whether it was adequate or we must go through the great labour of fitting entirely anew a parabola of a still higher order‡.

On the contrary it was fairly easy in the Variate Difference Method to test the adequacy of the order of differences taken and so determine whether we were approaching an elimination of the time-factor. If not, it was not so serious a matter to take still another difference. We take it that Mr Yule whatever method he would select for smoothing, would agree that the fundamental point is to correlate X and Y after such a smoothing process. This does not, however, seem to be the opinion of another critic, Dr Warren M. Persons§ of Harvard. He does admit that curve fitting may be preferable to taking "moving averages"—which procedure appears to the present writers a most fallacious manner of smoothing—but he demands that the smoothing curve shall "increase or decrease regularly according to some principle"§. This axiom, whatever may be its exact significance, allows him to accept a sloping straight line, or to consider two sloping straight lines as good as a second order parabola. Nevertheless he admits a second order parabola, or the "compound

* *Fifth International Congress of Mathematicians*, Cambridge, 1912 (II).

† *Tracts for Computers*, No. VI. Smoothing. Cambridge University Press.

‡ Tchebycheff's method to some extent surmounts this difficulty, but the computations are even more laborious than those of Least Squares.

§ *Quarterly Publications of the American Statistical Association*, Vol. XVI. June, 1917.

interest law" $x = a \,.\, C^t$. He seems, however, to have some objection to an nth order parabola form of smoothing; why, we cannot say, unless it is because it takes more labour to fit, or because for some reason or other he supposes it does not fall under his axiom. The secular trend of the general deathrate could certainly not be fitted by two straight lines, a second order parabola or the "compound interest law." In such a case it appears much better to take a high order parabola, but not so high as to introduce the sinuosities peculiar to the Sprague and King methods of osculatory graduation. It is very clear that we should get better results with 3rd to 5th order parabolae than Dr Persons does* with straight lines in the cases of Sauerbeck's Index Number of Wholesale Prices or the London Bank-Clearings.

What both Dr Persons and Mr Yule state, i.e. that we must be careful to see that our data give enough decimal places to cover our higher differences is, of course, correct. It is not a criticism of the method, but of application of the method beyond its proper range in the case of inadequate data. In matters like death- and birth-rates of large populations adequate decimal places can generally be obtained. This, however, may not be possible with some of the index numbers provided by various authorities.

On the other hand Dr Persons' statement, made after citing "Student," Dr Anderson, Miss Cave and one of ourselves, appears to us quite incorrect, namely: that "the writers on the Variate Difference correlation method all assume that 'the true r_{XY}' is for pairs concurrent in time" (*loc. cit.* p. 3).

Now Dr Persons' paper was issued in June 1917 and in May 1915 more than two years earlier the present authors published a paper in the number of *Biometrika* following that of Dr Anderson and the Cave-Pearson paper in which one of the essential features was the correlation of X_t with Y_{t+1}, Y_{t+2}, Y_{t+3} and Y_{t+4}. This paper was actually written at the same time as the Cave-Pearson paper and was not published till the following number on account of space. We were therefore quite conscious of the possibility of "lag." Indeed the possibility of lag had been considered many years before 1914 in correlating barometric heights on either side of the Atlantic by differences. What we overlooked was that the co-existence of these correlations might invalidate "Student's" assumptions.

This brings us to the main problem, the question of the assumptions made originally by "Student." They were—taking X_t and Y_t as the variates after removal of the secular trend—as follows:

X_t and $X_{t\pm\tau}$ are not correlated,
Y_t and $Y_{t\pm\tau'}$ are not correlated,
X_t and $Y_{t\pm\tau'}$ are not correlated.

The latter assumption will allow for the lag $t' - t$, if we merely replace it by saying X_t and $Y_{t'\pm\tau'}$ are not correlated, i.e. X_t and $Y_{t'}$ are supposed to be correlated. In fact all we are doing is shifting the Y curve backwards or forwards on the X curve.

* Figs. 9 and 10, *loc. cit.*

We pay therefore no attention to this phase of Dr Persons' criticism, because the possible existence of "lag" was realised long before Dr Persons wrote. Putting this aside there seem to us two main criticisms of the Variate Difference correlation method: Dr Persons' that we cannot assume that X is solely correlated with a single Y, and that the series of X's is not intercorrelated, nor the series of Y's. We think this is a valid criticism which has to be met, either by showing that there are many cases in which the causes which produce the X's and Y's do not last over more than one interval, or else by enlarging our method and supposing that correlations between the X's and Y's of the above character really exist. We think this can be done. But we will postpone its consideration until we have dealt with the second main criticism, that of Mr Yule. This criticism is the following one, namely: If there exists a short periodic term it will tend to dominate the whole investigation. In other words, if we are dealing with years as intervals, a two-year or a three-year period according to Mr Yule will of necessity swamp the non-periodic fluctuations.

Let us first examine the type of proof Mr Yule gives, and then elaborate the whole problem mathematically. A short cut can be made to the desired result by the use of central differences.

(2) Mr Yule starts from a periodic term, $x = A \sin(nt + \alpha)$, and assumes that this is what we have to deal with in investigations of this kind. Now we think this is very frequently incorrect, and that an error of a serious kind affects not only Mr Yule's reasoning, but a good deal of that of other workers on periodic analysis. The variate itself may well be of the form $A \sin(nt + \alpha)$, but in most cases we are not given values of the variate itself, but of its sum or integral for the unit of time, and it is this integral that we have to deal with in our range returns. Thus:

$$x_p = \int_{(p-1)h}^{ph} A \sin(nt + \alpha)\, dt = A \frac{\tau}{\pi} \sin \frac{\pi h}{\tau} \sin \left\{ \alpha + \frac{2\pi (p - \tfrac{1}{2}) h}{\tau} \right\} \quad \ldots (\text{i}),$$

where h is the subrange, $\tau = 2\pi/n$, the period.

Now $(p - \frac{1}{2})\, h$ is the time at the centre of the pth range h, and we can write with Mr Yule

$$x_p = A' \sin \left\{ \alpha + 2\pi (p - \tfrac{1}{2}) \frac{h}{\tau} \right\},$$

but to do so is to overlook the fact that this amplitude A' will as a rule contain a function of the period τ itself.

Next if we take central differences

$$\begin{aligned} \delta^2 x_p &= x_{p+1} + x_{p-1} - 2x_p \\ &= -A \frac{\tau}{\pi} \sin \frac{\pi h}{\tau}\, 2^2 \sin^2 \frac{\pi h}{\tau} \sin \left(\alpha + \frac{2\pi (p - \tfrac{1}{2}) h}{\tau} \right), \end{aligned}$$

and

$$\begin{aligned} \delta^{2q} x_p &= (-1)^q A \frac{\tau}{\pi} \sin \frac{\pi h}{\tau} \left(2^2 \sin^2 \frac{\pi h}{\tau} \right)^q \sin \left(\alpha + \frac{2\pi (p - \tfrac{1}{2}) h}{\tau} \right) \\ &= (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau} \right)^q x_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{ii}). \end{aligned}$$

Now let us consider the mean value of $\delta^{2q} x_p$ using square brackets here and throughout this paper to denote a mean value, i.e. $[z]$ = mean value of z.

Accordingly:

$$[\delta^{2q} x_p] = (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q [x_p]$$

$$= (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q A' \left[\sin\left\{\alpha + \frac{2\pi (p-\frac{1}{2})h}{\tau}\right\}\right]$$

$$= (-1)^q \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q \frac{A' \sin\left(\alpha + \frac{\pi\lambda h}{\tau}\right) \sin \frac{\pi\lambda h}{\tau}}{\lambda \sin \frac{\pi h}{\tau}} \quad \text{.....(iii),}$$

where λ is the number of intervals under consideration.

If λh be small as compared with τ, this expression will not tend to vanish with λ, but give a limit $\sin \alpha$. On the other hand if τ be small as compared with λh, the total range of observation, the term is of the order $1/\lambda$, and will tend to be negligible, i.e. for short period terms. Assuming then that we are dealing with "short period" terms we can now proceed to find the standard deviation of the qth central difference

$$[(\delta^{2q} x_p)^2] = \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^{2q} A'^2 \left[\sin^2\left(\alpha + \frac{2\pi (p-\frac{1}{2})h}{\tau}\right)\right] \quad \text{.....(iv),}$$

and we require to find the mean sum of squares of sines, or

$$\left[\frac{1}{2}\left\{1 - \cos\left(2\alpha + \frac{4\pi (p-\frac{1}{2})h}{\tau}\right)\right\}\right] = \frac{1}{2}\left\{1 - \frac{\cos\left(2\alpha + \frac{2\pi\lambda h}{\tau}\right) \sin \frac{2\pi\lambda h}{\tau}}{\lambda \sin \frac{2\pi h}{\tau}}\right\}.$$

Here the transcendental term will disappear for terms of short period and we have:

$$[(\delta^{2q} x_p)^2] = \frac{1}{2}\left(2^2 \sin^2 \frac{\pi h}{q}\right)^{2q} A^2 \frac{\tau^2}{\pi^2} \sin^2 \frac{\pi h}{\tau} \quad \text{.....(v).}$$

Lastly, let us find the correlation between two periodic terms, say x_p and $y_p = B \sin(n't + \beta)$. Here y_p refers only to the pth interval of the y variate, which may or may not be concurrent with the pth interval for x. Let $n' = 2\pi/\tau'$. Then as before for "short periods"

$$[(\delta^{2q} y_p)^2] = \frac{1}{2}\left(2^2 \sin^2 \frac{\pi h}{\tau'}\right)^{2q} B^2 \frac{\tau'^2}{\pi^2} \sin^2 \frac{\pi h}{\tau'} \quad \text{.....(vi),}$$

$$[\delta^{2q} x_p \, \delta^{2q} y_p] = \left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^q \left(2^2 \sin^2 \frac{\pi h}{\tau'}\right)^q AB \frac{\tau\tau'}{\pi^2} \sin \frac{\pi h}{\tau} \sin \frac{\pi h}{\tau'}$$

$$\times \left[\sin\left(\alpha + \frac{2\pi (p-\frac{1}{2})h}{\tau}\right) \sin\left(\beta + \frac{2\pi (p-\frac{1}{2})h}{\tau'}\right)\right].$$

First suppose τ and τ' to be different. Then we have to find:

$$\left[\sin\left(\alpha + \frac{2\pi (p-\frac{1}{2})h}{\tau}\right) \sin\left(\beta + \frac{2\pi (p-\frac{1}{2})h}{\tau'}\right)\right],$$

which may be expressed as:

$$= \frac{1}{2}\left\{\left[\cos(\alpha-\beta)+2\pi h\,(p-\tfrac{1}{2})\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)\right]-\left[\cos\left(\alpha+\beta+2\pi h\,(p-\tfrac{1}{2})\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)\right)\right]\right\}$$

$$= \frac{1}{2\lambda}\left\{\frac{\cos\left\{\alpha-\beta+\pi\lambda h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)\right\}\sin\pi h\lambda\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}{\sin\pi h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}\right.$$

$$\left.-\frac{\cos\left\{\alpha+\beta+\pi\lambda h\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)\right\}\sin\pi h\lambda\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)}{\sin\pi h\left(\frac{1}{\tau}+\frac{1}{\tau'}\right)}\right\}.$$

If the periods be short the $\frac{1}{\lambda}$ term makes this very small. The exception is when $\tau=\tau'$, then

$$\frac{\sin\pi h\lambda\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}{\lambda\sin\pi h\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)}$$

takes the value unity and we have

$$[\delta^{2q}x_p\,\delta^{2q}y_p]=\left(2^2\sin^2\frac{\pi h}{\tau}\right)^q\left(2^2\sin^2\frac{\pi h}{\tau'}\right)^q AB\frac{\tau\tau'}{\pi^2}\sin\frac{\pi h}{\tau}\sin\frac{\pi h}{\tau'}\times\tfrac{1}{2}\cos(\alpha-\beta)\quad\text{(vii)},$$

the second term being negligible compared to this.

Accordingly the correlation between two short period terms of equal period is simply

$$\cos(\alpha-\beta)\quad\text{.....(viii)},$$

or perfect if there be no difference in phase, zero if $\frac{1}{4}$ period difference in phase, and perfect but of negative sign if $\frac{1}{2}$ period difference in phase. The result is independent of the difference used if we work with central differences. If therefore the only term were a single short period term the correlation of successive central differences ought to give us a constant correlation. If we find this correlation negligible we are driven to the conclusion that there is no such single short period term in existence common to both variates or that there must be $\frac{1}{4}$ period difference in phase.

Turn now to the theory of non-periodic fluctuations. We know that if X and Y be the non-periodic fluctuations on the theory of "Student"

$$\left.\begin{aligned}[(\delta^{2q}X_p)^2]&=\frac{(4q)!}{(2q)!(2q)!}\sigma_X{}^2\\ [(\delta^{2q}Y_p)^2]&=\frac{(4q)!}{(2q)!(2q)!}\sigma_Y{}^2\end{aligned}\right\}\quad\text{.....(ix)},$$

$$[\delta^{2q}X_p\delta^{2q}Y_p]=\frac{(4q)!}{(2q)!(2q)!}\sigma_X\sigma_Y r_{XY}\quad\text{.....(x)},$$

and accordingly

r_{XY} = correlation of $\delta^{2q}x_p$ and $\delta^{2q}y_p$,

without consideration of removing secular trend. This is true if only *one* X is correlated with *one* Y—it is not needful to suppose these values of X and Y to be concurrent as Dr Persons asserts those who used the method have supposed. On the other hand if the series of X's and the series of Y's and again the series of X, Y's have correlation, we must introduce factors, which we will call for a moment $\phi(\rho')$, $\phi(\rho'')$ and $\phi(\rho)$, into these results

$$\left.\begin{aligned}[(\delta^{2q}X_p)^2] &= \frac{(4q)!}{(2q)!(2q)!}\phi(\rho')\sigma_X^2 \\ [(\delta^{2q}Y_p)^2] &= \frac{(4q)!}{(2q)!(2q)!}\phi(\rho'')\sigma_Y^2\end{aligned}\right\} \quad \text{.....(xi),}$$

$$[(\delta^{2q}X_p\delta^{2q}Y_p)] = \frac{(4q)!}{(2q)!(2q)!}\phi(\rho)\sigma_X\sigma_Y r_{XY} \quad \text{.....(xii),}$$

and we shall have

$$r_{XY} = \frac{\sqrt{\phi(\rho')\,\phi(\rho'')}}{\phi(\rho)} \times \text{correlation of } \delta^{2q}x_p \text{ and } \delta^{2q}y_p \quad \text{.....(xiii).}$$

We will discuss these factors shortly.

(3) Supposing, however, "Student's" theory to hold, let us investigate what happens if we have a combination of secular trend, periodic terms and non-periodic fluctuations:

$$\left.\begin{aligned}x_p &= a_0 + a_1 t + \ldots + a_{n-1}t^{n-1} + S_\tau\left\{A_\tau' \sin\left(\alpha_\tau + \frac{2\pi}{\tau}(p-\tfrac{1}{2})h\right)\right\} + X_p, \\ y_p &= b_0 + b_1 t + \ldots + b_{m-1}t^{m-1} + S_\tau\left\{B_\tau' \sin\left(\beta_\tau + \frac{2\pi}{\tau}(p-\tfrac{1}{2})h\right)\right\} + Y_p,\end{aligned}\right\} \quad \text{...(xiv).}$$

Or, if $2q$ be greater than n and m,

$$[\delta^{2q}x_p\delta^{2q}y_p] = S_\tau\left\{\tfrac{1}{2}A_\tau' B_\tau'\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\cos(\alpha_\tau - \beta_\tau)\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_X\sigma_Y r_{x_p y_p} \text{...(xv),}$$

$$\left.\begin{aligned}[(\delta^{2q}x_p)^2] &= S_\tau\left\{\tfrac{1}{2}A_\tau'^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_X^2 \\ [(\delta^{2q}y_p)^2] &= S_\tau\left\{\tfrac{1}{2}B_\tau'^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\} + \frac{(4q)!}{(2q)!(2q)!}\sigma_Y^2\end{aligned}\right\} \quad \text{.....(xvi).}$$

Now we must here make a two-fold confession: first that one of us actually queried in proof Dr Anderson's statement in *Biometrika*, Vol. x. p. 279, about the elimination of periodic terms, and took the earliest opportunity when he did not modify it to express disagreement with what we thought to be his views (*Biometrika*, Vol. x. p. 503 footnote). In that place we also say that: "Should a more extended experience show that there is a real, if slight *positive* correlation between deathrates at three years' interval, while there is considerable *negative* correlation at one and two years' interval we should be compelled to discuss whether there is not something periodic in the nature of the heavy and light deathrates of infancy and childhood." In view of that statement it is almost absurd for Mr Yule to suggest that we overlooked the possibility of a short period term, yet he actually appears to be slightly annoyed because we inserted this footnote! But that footnote continues: "We have been unable to trace any sign of such periodicity either in the deathrate

or the graphs drawn, but we do not believe that a very short periodicity would be eliminated by the variate difference method using any moderate number of differences." This appears to have been also Dr Anderson's view throughout, only both Mr Yule and we misread his statements. We overlooked, perhaps with some justification, his qualifying phrase: "Ja mehr noch, man kann beweisen, daß überhaupt aller mehr oder minder 'glatten Reihen,' alle bei denen eine genügende positive Korrelation zwischen den Nachbargliedern bemerkbar ist, für die Praxis beim endlichen Differenzieren verschwinden" (*Biometrika*, Vol. x. p. 279). The short period terms demand a *negative* correlation. As a matter of fact as we shall indicate later, no short periodic terms of any importance exist in the material used by us in our first paper; our original opinion is confirmed by more ample examination. But we must here confess our second point: we were actually indifferent to the short period terms if such existed, we felt comparatively certain of their insignificance in our material, and further, if they actually were present, they seemed to us just as likely to be a product of the organic relationship we were seeking, as a complete mask of it, which we suppose to be Mr Yule's view. It is possible, if such terms exist, that they form a reasonable element in the correlation of X_p and Y_p after removal of the secular trend. What we were concerned with was to get out of the morass which Sir Arthur Newsholme, Mortara and other statisticians had got us into, by correlating time growths of various factors and interpreting the resulting intense correlations as causal factors. That end seemed to us achieved by the variate difference method and therefore we praised it highly as breaking new ground.

Mr Yule criticises us and says (i) he does not believe in the existence of fluctuating variations and (ii) if such existed they would be swamped by differencing and the shortest period terms would alone survive. He postulates the existence of the latter. His processs of demonstrating appears to us remarkable. He notes the relative size of the multiplying factors on differencing, i.e.

$$\left(2^2 \sin^2 \frac{\pi h}{\tau}\right)^{2q} \quad \text{and} \quad \frac{(4q)!}{(2q)!(2q)!},$$

and shows that the former will be the greater for short period terms. Now the former will be the greater when

$$\sin \frac{\pi h}{\tau} > \frac{1}{2} \left(\frac{(4q)!}{(2q)!(2q)!}\right)^{\frac{1}{4q}}.$$

For second differences this gives $\tau < 3{\cdot}496h$,
„ fourth „ „ „ $< 3{\cdot}090h$,
„ sixth „ „ „ $< 2{\cdot}901h$,
„ eighth „ „ „ $< 2{\cdot}787h$,
„ tenth „ „ „ $< 2{\cdot}708h$.

It is unnecessary to go further because it is unlikely that we could go beyond 10th differences owing to the imperfection of our data, or the great labour of computation.

Now we think there is an error in Mr Yule's reasoning. For the variate difference investigation X and Y are the whole fluctuations above the secular trend. Mr Yule compares σ_X with A_τ', that is to say he compares the standard deviation of one portion of his fluctuation with the amplitude of another portion! But at least he ought to have compared $\frac{1}{\sqrt{2}} A_\tau'$ with σ_Y, and even this is not legitimate.

The fluctuation is

$$S_\tau A_\tau' \sin\left\{\alpha_\tau + \frac{2\pi}{\tau}(p - \tfrac{1}{2})\, h\right\} + X_p,$$

the S.D.^{2} of the whole is

$$S_\tau (\tfrac{1}{2} A_\tau'^2) + \sigma_X^2,$$

and accordingly σ_X^2 should be compared with $S_\tau(\frac{1}{2}A_\tau'^2)$ and not with $\frac{1}{2}A_\tau'^2$, still less with $(A_\tau')^2$ as Mr Yule compares it.

Does, however, $\frac{1}{2}A_\tau'^2$ necessarily form the largest term in the summation? We see no reason for supposing it in the least likely.

$$\tfrac{1}{2}A_\tau'^2 = \tfrac{1}{2}A_\tau^2 \frac{\tau^2}{\pi^2} \sin^2 \frac{\pi h}{\tau},$$

and this is not a maximum when τ is least but when

$$\sin \frac{\pi h}{\tau} \Big/ \frac{\pi h}{\tau}$$

is a maximum, i.e. when $\sin z/z$ is a maximum, or when $z = \tan z$.

In fact if we evaluate the function $\frac{\tau^2}{h^2} \sin^2\left(\frac{\pi h}{\tau}\right)$, we have the series of values in the following table. They show that *for equal amplitudes* the short period terms contribute *least* to the S.D.^{2} of the fluctuations, and that we cannot straight off assert as Mr Yule has done that the low period terms of less than 3 years will dominate the result reached.

$\tau = 1$	$\frac{\tau^2}{h^2}\sin^2\left(\frac{\pi h}{\tau}\right)$	τ	$\frac{\tau^2}{h^2}\sin^2\left(\frac{\pi h}{\tau}\right)$
h	0·000	$10h$	9·549
$2h$	4·000	$12h$	9·646
$3h$	6·750	$18h$	9·786
$4h$	8·000	$25h$	9·818
$5h$	8·637	$30h$	9·834
$6h$	9·000	$40h$	9·849
$7h$	9·225	$50h$	9·855
$8h$	9·373	$100h$	9·866
$9h$	9·475	$150h$	9·868

Now it seems to us that we have to take for any comparison at all some ratio between σ_X and the total S.D. due to periodic terms, or we may take

$$\sigma_X^2 = m^2 S_\tau \left(\tfrac{1}{2} A_\tau^2 \frac{\tau^2}{\pi^2} \sin^2 \frac{\pi h}{\tau}\right),$$

and if we assume any one value to be more important here than another, it will not be the term of *lowest* period. In any case it is not needful to consider that the term of lowest period contributes most to the S.D. of the fluctuations.

Now return to the S.D.^{2} of the $2q$th difference. We have to consider the parts of the expression

$$S_\tau\left\{\tfrac{1}{2}A_\tau{}^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\}+\sigma_x{}^2\frac{(4q)!}{(2q)!(2q)!}.$$

Accordingly if we assume $m^2=1$ or the random terms and the periodic of the same order, we have to investigate the relative order of

$$\frac{S_\tau\left\{\tfrac{1}{2}A_\tau{}^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q}\right\}}{S_\tau\left\{\tfrac{1}{2}A_\tau{}^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\right\}} \text{ and } \frac{(4q)!}{(2q)!(2q)!} \quad \text{.....(xvii),}$$

and in considering only

$$\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q} \text{ and } \frac{(4q)!}{(2q)!(2q)!}$$

to obtain a dominant term Mr Yule has omitted to consider the multiplier

$$\tfrac{1}{2}A_\tau{}^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\Big/S_\tau\left\{\tfrac{1}{2}A_\tau{}^2\left(\frac{\tau^2}{\pi^2}\sin^2\frac{\pi h}{\tau}\right)\right\}.$$

We cannot reduce this to unity because we have just seen that the lowest τ term is not necessarily the dominant term in the denominator.

We have to evaluate $\left(\frac{\tau}{h}\right)^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q+1}$ for various values of τ/h and q to discover where we are. We will do this for second, fourth, sixth, eighth and tenth differences, and for $\tau=2h$, $3h$, $4h$, $6h$, $8h$ and $10h$, placing also on record the value of $(4q)!/\{(2q)!(2q)!\}$. We have the following table:

Table of $\left(\frac{\tau}{h}\right)^2\left(2^2\sin^2\frac{\pi h}{\tau}\right)^{2q+1}$.

$\tau/h=$	2	3	4	6	8	10	$(4q)!/\{(2q)!(2q)!\}$
$q=1$	256	243	128	36	12·87	5·57	6
2	4096	2187	512	36	4·41	0·81	70
3	65536	19683	2048	36	1·51	0·12	924
4	1048576	177147	8192	36	0·52	0·02	12870
5	16777216	1594323	32768	36	0·17	0·00	184756
$q=0$	16	27	32	36	37·49	38·20	—

Now it is clear that if we suppose only *one* short periodic term to occur, it will have no influence at all, if the period be four times or more the subrange of tabling. In other words unless the non-periodic fluctuation was very small as compared to the periodic all trace of the periodic term would have disappeared by the 8th or 10th differencing. Anderson would therefore have been justified in his statement

as we at first read it had he confined it to periodic terms of four or more times the subrange.

When we turn to terms of three times the subrange if there was only one periodic term, its order would at the 8th be in the ratio of 10 to 20, and at the 10th difference in the ratio of 10 to 31. Such a term therefore would not swamp the non-periodic fluctuation; it would in fact be considerably smaller than it. If it were combined with two or three other terms of longer period but about the same or larger amplitude, it would be swamped by the non-periodic fluctuation.

Now turning to the term where $\tau/h = 2$, we notice that if only a single such term existed, its influence at the tenth difference would be as 6 : 1, at the eighth difference as about 5 : 1 and at the sixth difference as about 4 : 1 relative to the non-periodic fluctuation. If it was associated with two or three other terms of longer periodicity, its influence would be somewhat *less* than that of the non-periodic term. What is clear is that even here it would be exaggeration to talk of swamping by the term of short period. The contributions of the two terms are much of the same order, and which will dominate in the S.D. of the fluctuations will depend on how far those fluctuations are due to a series of periodic terms, or to the non-periodic part of the fluctuations.

Further the actual correlation in $\delta^{2q}x$ and $\delta^{2q}y$ depends not only on the S.D.'s of the fluctuations, but on the correlations, on whether the periodic terms in x and y are, or are not, in the same phase.

It is therefore not really a case of one or other factor being swamped. The process removes the secular trend and correlates the residuals. In these residuals the factors which contribute chiefly to the correlation are the non-periodic and very short periodic terms if the latter exist. If they exist it would indicate that the residuals have a common factor, and that seems to me the very point we set out to inquire into. If one has a two element period and the other has not, that will lessen the correlation between the non-periodic fluctuations, but that is surely what we should anticipate.

On the whole it does not seem to us that Mr Yule's criticisms really lead one far, especially when he concludes them by stating that the time problem is that of discussing the relations "between oscillations of different durations, such oscillations being in all probability not strictly periodic, but up and down movements of greater or less rapidity" (p. 524). If they are not periodic, why confuse the issue by endeavouring to show that *true* periodic terms would swamp the residuals which he holds are not *true* periodic terms?

If terms of very small period exist, then those who desire to do so can find and eliminate them.

(4) The criticism that is raised by Persons that we cannot assume $S(X_p X_{p+s}) = 0$ seems to us more valid, although it is certainly not demonstrated if we test its validity by finding the X's from a couple of combined straight lines.

We think, however, that it is possible to meet this criticism. In the first place it can be met by simply taking out the secular trend by adequate smoothing and

correlating the residuals in as many ways as seem desirable. But theoretically it is also possible to learn something about these correlations by the variate-difference method, i.e. by the aid of the functions $\phi(\rho)$, $\phi(\rho')$, $\phi(\rho'')$ already referred to. This we now proceed to discuss.

Let us suppose the correlation of X_p, Y_{p+s} to be ρ_s and of X_p Y_{p-s} to be ρ_{-s}. Let us call ρ_s' the correlation of X_p and X_{p+s}, while ρ'_{-s} is that of X_p and X_{p-s}. Lastly ρ_s'' is the correlation of Y_p and Y_{p+s}, ρ''_{-s} of Y_p and Y_{p-s}. Then we readily find

$$\frac{[\Delta^n X_p \Delta^n Y_p]}{\sigma_X \sigma_Y} = \frac{(2n)!}{n!\,n!}\left\{(-1)^n \frac{n!\,n!}{(2n)!}\rho_{-n} + \ldots + \frac{n(n-1)}{(n+1)(n+2)}\rho_{-2} - \frac{n}{n+1}\rho_{-1} + \rho_0 - \frac{n}{n+1}\rho_1 + \frac{n(n-1)}{(n+1)(n+2)}\rho_2 - \ldots (-1)^n \frac{n!\,n!}{(2n)!}\rho_n\right\}$$

$$= \frac{(2n)!}{n!\,n!}\phi(n, \rho_0, \rho) \text{ say} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xviii).}$$

Similarly $$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_p]}{\sigma_X \sigma_Y} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}\phi(n+1, \rho_0, \rho) \quad \ldots\ldots\ldots\text{(xix),}$$

and accordingly* $$\frac{[\Delta^{n+1} X_p \Delta^{n+1} Y_p]}{[\Delta^n X_p \Delta^n Y_p]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, \rho_0, \rho)}{\phi(n, \rho_0, \rho)} \quad \ldots\ldots\ldots\ldots\text{(xx).}$$

Again

$$\frac{[(\Delta^n X_p)^2]}{\sigma_X^2} = \frac{(2n)!}{n!\,n!}\left\{(-1)^n \frac{n!\,n!}{(2n)!}\rho'_{-n} + \ldots + \frac{n(n-1)}{(n+1)(n+2)}\rho'_{-2} - \frac{n}{n+1}\rho'_{-1} + 1 - \frac{n}{n+1}\rho_1' + \frac{n(n-1)}{(n+1)(n+2)}\rho_2' - \ldots (-1)^n \frac{n!\,n!}{(2n)!}\rho_n'\right\}$$

$$= \frac{(2n)!}{n!\,n!}\phi(n, 1, \rho') \text{ say} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxi),}$$

and accordingly $$\frac{[(\Delta^{n+1} X_p)^2]}{[(\Delta^n X_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, 1, \rho')}{\phi(n, 1, \rho')} \quad \ldots\ldots\ldots\ldots\text{(xxii).}$$

Similarly $$\frac{[(\Delta^{n+1} Y_p)^2]}{[(\Delta^n Y_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{\phi(n+1, 1, \rho'')}{\phi(n, 1, \rho'')} \quad \ldots\ldots\ldots\ldots\text{(xxiii).}$$

Now it is clear that our three equations (xx), (xxi) and (xxii) separate off the ρ_s, the ρ_s' and ρ_s'', and accordingly we can confine our attention to any one type in considering the solution. Further, by putting $\tilde{\rho}_s = \rho_s/\rho_0$ we can write

$$\phi(n+1, \rho_0, \rho) = \rho_0 \phi(n+1, \tilde{\rho})$$

and drop the second variable (ρ_0 or 1) out.

After finding $\tilde{\rho}$, ρ' and ρ'' we shall have finally to determine ρ_0 from

$$\frac{[\Delta^n X_p \Delta^n Y_p]}{\sqrt{[(\Delta^n X_p)^2][(\Delta^n Y_p)^2]}} = \rho_0 \frac{\phi(n, \tilde{\rho})}{\sqrt{\phi(n, \rho')\,\phi(n, \rho'')}} \quad \ldots\ldots\ldots\text{(xxiv).}$$

We may therefore leave ρ_0 out of account and consider we have to find the three series $\tilde{\rho}$, ρ', ρ''.

* We owe results (xviii) and (xxi) to Mr E. S. Pearson, who had used them in a thesis presented in 1921 and since published in *Biometrika*, Vol. XIV. pp. 37—39.

If we call the three observed quantities*

$$\tilde{Y}_{n,0}\left(4-\frac{2}{n+1}\right), \quad Y'_{n,0}\left(4-\frac{2}{n+1}\right), \quad Y''_{n,0}\left(4-\frac{2}{n+1}\right),$$

we have the three type equations

$$\tilde{Y}_{n,0}=\frac{\phi(n+1,\tilde{\rho})}{\phi(n,\tilde{\rho})}, \quad Y'_{n,0}=\frac{\phi(n+1,\rho')}{\phi(n,\rho')}, \quad Y''_{n,0}=\frac{\phi(n+1,\rho'')}{\phi(n,\rho'')},$$

to find the series $\tilde{\rho}$, ρ', ρ'' respectively. Now at first sight it might seem that all we have to do is to write down equations for enough values of n till we have adequate relations to find, say, the $\tilde{\rho}$'s, starting n after we have gone to a difference adequate to remove the secular trend. This would not, however, be possible, for each fresh equation introduces two more ρ's. Thus in $\tilde{Y}_n$ we have $\tilde{\rho}_{-(n+1)} \cdots \tilde{\rho}_{-1}\, \tilde{\rho}_1 \cdots \tilde{\rho}_{n+1}$, but in $\tilde{Y}_{n+1}$ we introduce $\tilde{\rho}_{-(n+2)}$ and $\tilde{\rho}_{n+2}$ in addition. Thus every new equation gives us two further unknowns. At the same time these unknowns will probably be very small and multiplied by very small coefficients of the order $n!\,n!/(2n!)$, i.e. the inverses of the terms in the last column of the table on p. 292. Remembering that the ρ's will most probably be rapidly convergent also, we see that if we suppose the secular term disposed of by the fourth difference, we might well neglect the terms involving $\tilde{\rho}_4$ and $\tilde{\rho}_{-4}$. This would leave us with six unknowns to be determined, $\tilde{\rho}_{-3}$, $\tilde{\rho}_{-2}$, $\tilde{\rho}_{-1}$, $\tilde{\rho}_1$, $\tilde{\rho}_2$, $\tilde{\rho}_3$, or we should need six equations, or the value of $\tilde{Y}_n$ from $n=4$ to 9 involving up to 10th differences would suffice. Theoretically the problem is solved, if we can be content with the correlations of seven adjacent fluctuations. These indeed if we found them would usually be quite adequate to determine the problem of "lag," and throw light by their rate of decay on Mr Yule's view as to "not strictly periodic oscillations." If "Student's" original hypotheses are correct, we should find all the ρ's but one sensibly zero, and this one would indicate the lag.

Now this method depends for safety on our accuracy in the high differences, and this can by no means always be attained. Accordingly we ask whether additional correlations cannot be found which will avoid the risk of too high differences. We have clearly still available the correlations of $\Delta^n X_p$ and $\Delta^n Y_{p+s}$. For example:

$$\frac{[\Delta^n X_p\, \Delta^n Y_{p+1}]}{\sigma_X \sigma_Y}=\frac{(2n)!}{n!\,n!}\left\{(-1)^n\frac{n!\,n!}{(2n)!}\rho_{-(n-1)}+\cdots\right.$$
$$+\frac{n(n-1)}{(n+1)(n+2)}\rho_{-1}-\frac{n}{n+1}\rho_0+\rho_1-\frac{n}{n+1}\rho_2+\frac{n(n-1)}{(n+1)(n+2)}\rho_3\cdots$$
$$\left.+(-1)^n\frac{n!\,n!}{(2n)!}\rho_{n+1}\right\} \quad \text{.....(xxv),}$$

$$\frac{[\Delta^n X_p\, \Delta^n Y_{p-1}]}{\sigma_X \sigma_Y}=\frac{(2n)!}{n!\,n!}\left\{(-1)^n\frac{n!\,n!}{(2n)!}\rho_{-(n+1)}+\cdots\right.$$
$$+\frac{n(n-1)}{(n+1)(n+2)}\rho_{-3}-\frac{n}{n+1}\rho_{-2}+\rho_{-1}-\frac{n}{n+1}\rho_0+\frac{n(n-1)}{(n+1)(n+2)}\rho_1\cdots$$
$$\left.+(-1)^n\frac{n!\,n!}{(2n)!}\rho_{n-1}\right\} \quad \text{.....(xxvi).}$$

* The subscript zero is introduced into Y to mark that we are correlating concurrent $\Delta^n X_p$ and $\Delta^n Y_p$, not $\Delta^n X_p$ and $\Delta^n Y_{p+s}$. See p. 285.

These non-symmetrical functions will not converge as rapidly as the symmetrical, but even if we had to take an additional pair of ρ's (for example $\tilde{\rho}_{-4}$ and $\tilde{\rho}_4$) into account, we could get eight equations for finding them as well as an additional control equation by using only results up to the seventh differences, supposing the secular trend to disappear at the fourth.

As before, if $\tilde{\rho}_s = \rho_s/\rho_0$ and supposing E to be the usual finite difference operator acting on ρ_s but not on ρ_0, we have:

$$\frac{[\Delta^{n+1} X_p \, \Delta^{n+1} Y_{p+1}]}{[\Delta^n X_p \, \Delta^n Y_{p+1}]} = \left(4 - \frac{2}{n+1}\right) \frac{E\phi(n+1, \tilde{\rho})}{E\phi(n, \tilde{\rho})},$$

or if the known left-hand side be $\tilde{Y}_{n,1}\left(4 - \frac{2}{n+1}\right)$,

$$\tilde{Y}_{n,1} = \frac{E\phi(n+1, \tilde{\rho})}{E\phi(n, \tilde{\rho})} \quad \text{.....(xxvii).}$$

Similarly
$$\tilde{Y}_{n,-1} = \frac{E^{-1}\phi(n+1, \tilde{\rho})}{E^{-1}\phi(n, \tilde{\rho})} \quad \text{.....(xxviii).}$$

Similar equations:

$$Y'_{n,1} = \frac{E\phi(n+1, \rho')}{E\phi(n, \rho')}, \quad Y'_{n,-1} = \frac{E^{-1}\phi(n+1, \rho')}{E^{-1}\phi(n, \rho')},$$

$$Y''_{n,1} = \frac{E\phi(n+1, \rho'')}{E\phi(n, \rho'')}, \quad Y''_{n,-1} = \frac{E^{-1}\phi(n+1, \rho'')}{E^{-1}\phi(n, \rho')} \quad \text{.....(xxix),}$$

will help to give the ρ_s' and ρ_s'' respectively.

The above results will probably reduce our working, according to the nature of the secular trend, to something like 6th or 7th differences. Only practical experience can show how far the equations proposed are sensitive enough to give useful results. We propose shortly to test them by actually subtracting the secular trend and correlating the residuals, and also by finding the correlations by the present indirect method.

It will be observed of course that any ρ-system marks the rate of decadence of the correlated series of variates. For example ρ'_{-s} is the correlation of X_{p-s} with X_p and ρ'_{+s} of X_p with X_{p+s}. It would be no unreasonable assumption to suppose $\rho'_{-s} = \rho'_{+s}$ and this would reduce our requisite equations by nearly one half. The assumption that an oscillation can be represented by a harmonic term involves such a relationship. By retaining, however, the distinction it may be possible to allow for something of the nature of an asymmetrical oscillation.

One case is, however, of special interest: if $\rho_s' = \tilde{\rho}_s = \rho_s''$ for all values of s, then the correlation of $\Delta^n X_p$ and $\Delta^n Y_p$ is equal to the correlation of X and Y. That is we have "Student's" hypothesis replaced by a much more general one, namely the hypothesis that the correlation between the variates dies out at the same rate. This marks, we think, a considerable extension of the legitimate range of application of the simple Variate Difference Method. Let us examine the exact meaning of the needful relations. We have $\rho_s' = \tilde{\rho}_s$ or $\rho_s'\rho_0 - \rho_s = 0$, that is:

correlation of X_p, Y_{p+s} = correlation of X_p, $Y_p \times$ correlation of X_p, X_{p+s},

but the correlation of X_p and Y_p is that of X_{p+s} and Y_{p+s}. Hence correlation of X_p and Y_{p+s} for constant X_{p+s} is zero. Similarly the correlation of Y_p and X_{p+s} for constant X_{p+s} is zero. In other words there is no relation between any non-corresponding (we do not say non-concurrent because we may allow for "lag") X and Y, except indirectly through the X which corresponds to the Y. If we make the X which corresponds to any Y constant, then that Y will have zero correlation with all other X's. Y is indirectly affected by non-corresponding X's owing to its correlation with its own X's *.

We can reach, however, a still more general hypothesis. Let us suppose the three sets of correlations to degrade not at the same rate but at different rates, i.e. let

$$\rho_s = \epsilon^s \rho_0, \quad \rho_s' = \epsilon'^s, \quad \rho_s'' = \epsilon''^s.$$

We will call $\phi(n, \epsilon)$ the series

$$\left((-1)^n \frac{n!\,n!}{(2n)!}\epsilon^n + \ldots + \frac{n(n-1)}{(n+1)(n+2)}\epsilon^2 - \frac{n}{n+1}\epsilon + 1\right).$$

Then we have

$$\frac{[\Delta^{n+1}x_p\,\Delta^{n+1}y_p]}{[\Delta^n x_p\,\Delta^n y_p]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon) - 1}{2\phi(n, \epsilon) - 1} \quad \ldots\ldots\ldots\ldots\text{(xxx)},$$

$$\frac{[(\Delta^{n+1}x_p)^2]}{[(\Delta^n x_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon') - 1}{2\phi(n, \epsilon') - 1} \quad \ldots\ldots\ldots\text{(xxxi)},$$

$$\frac{[(\Delta^{n+1}y_p)^2]}{[(\Delta^n y_p)^2]} = \left(4 - \frac{2}{n+1}\right)\frac{2\phi(n+1, \epsilon'') - 1}{2\phi(n, \epsilon'') - 1} \quad \ldots\ldots\ldots\text{(xxxii)},$$

and since the left-hand sides are known we can find from them ϵ, ϵ' and ϵ''. Mr Henderson has kindly formed for us a table of the function on the right-hand side for $n = 1$ to 10 and ϵ from -1 to $+1$ by tenths †. It is possible by interpolation from this table to find the ϵ's readily. When they are found we have for the correlation of $\Delta^n x_p\,\Delta^n y_p$,

$$r_{\Delta^n x_p \Delta^n y_p} = \frac{\{2\phi(n, \epsilon) - 1\}\, r_{XY}}{\sqrt{2\phi(n, \epsilon') - 1}\,\sqrt{2\phi(n, \epsilon'') - 1}},$$

or

$$r_{XY} = r_{\Delta^n x_p \Delta^n y_p}\frac{\sqrt{2\phi(n, \epsilon') - 1}\,\sqrt{2\phi(n, \epsilon'') - 1}}{2\phi(n, \epsilon) - 1} \quad \ldots\ldots\text{(xxxiii)}.$$

We can test the goodness of the hypothesis by seeing if the values of ϵ, ϵ and ϵ'' agree as determined from successive values of n.

If ϵ be positive then $2\phi(n, \epsilon) - 1$ always decreases with increase of n or increase of ϵ. Now we should suppose as a rule that the degradation of the correlation of the X's among themselves, or of the Y's among themselves, would be less rapid than that of X and Y. Accordingly it follows that in such cases

$$r_{XY} < r_{\Delta^n x_p \Delta^n y_p},$$

* This is the Mendelian view of gametic constitution. There is a correlation between grandparents and grandchild, but if you fix the constitution of the parents this correlation vanishes.

† See Appendix to this paper, p. 310.

or the value obtained by "Student" is a superior limit to the correlation of the fluctuations from the secular trend. If on the other hand ϵ' and ϵ'' are negative, as would almost certainly be the case if there were a large two interval period, then r_{XY} will be larger than the value assigned by "Student," on the same hypothesis of degradation. In any case it appears to us that to calculate the ϵ's by aid of Mr Henderson's table will throw light on the inter-relation of the fluctuations from the secular trend, even if it be only a step preliminary to a fuller determination of the ρ's.

We think the theory of the variate difference method thus generalised will meet such criticisms as those of Persons, which are valid. It may be harder to meet those of Mr Yule who after attempting to bludgeon the whole business with a two interval harmonic period, then turns round and states that such harmonic periods do not occur but up and down movements of greater or less rapidity, these movements from the secular trend, for some unprovided reason, not being identical with the non-periodic fluctuations which the believers in the variate difference method were attempting to study, nor with the "mehr oder minder glatten Reihen" which Anderson asserts disappear by differencing.

(5) Let us now turn to the case of a single periodic term in our variates and see how it will influence our equations.

The term in X_p being

$$A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha+2\pi\left(p-\tfrac{1}{2}\right)\frac{h}{\tau}\right\},$$

that in X_{p+s} will be

$$A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha+2\pi\left(p+s-\tfrac{1}{2}\right)\frac{h}{\tau}\right\}$$

$$=A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha+\frac{2\pi sh}{\tau}+2\pi\left(p-\tfrac{1}{2}\right)\frac{h}{\tau}\right\},$$

and accordingly the correlation between X_p and X_{p+s} will be

$$\rho'_s=\cos\frac{2\pi sh}{\tau}=\rho'_{-s} \quad \text{.....(xxxiv),}$$

or there should be the same correlation between X_p and X_{p+s} or X_p and X_{p-s}.

Let us write $\gamma=2\pi h/\tau$; then turning to (xxi)

$$\phi(n,1,\rho')=2\left\{1-\frac{n}{n+1}\rho_1'+\frac{n(n-1)}{(n+1)(n+2)}\rho_2'-\dots\right.$$

$$\left.+(-1)^n\frac{n!\,n!}{(2n)!}\rho_n'\right\}-1$$

$$=2\phi(n,\gamma)-1 \quad \text{.....(xxxv),}$$

where

$$\phi(n,\gamma)=1-\frac{n}{n+1}\cos\gamma+\frac{n(n-1)}{(n+1)(n+2)}\cos 2\gamma-\dots$$

$$+(-1)^n\frac{n!\,n!}{(2n)!}\cos n\gamma \quad \text{.....(xxxvi).}$$

Hence
$$2\phi(n,\gamma)-1=\frac{n!\,n!}{(2n)!}\left(2\sin\frac{\gamma}{2}\right)^{2n} \quad \text{.....(xxxvii),}$$

and
$$\left(4-\frac{2}{n+1}\right)\frac{2\phi(n+1,\gamma)-1}{2\phi(n,\gamma)-1}=2(1-\cos\gamma).$$

Accordingly
$$\frac{[(\Delta^{n+1}X_p)^2]}{[(\Delta^n X_p)^2]}=2(1-\cos\gamma) \quad \text{.....(xxxviii),}$$

or
$$\sin\frac{\pi h}{\tau}=\tfrac{1}{2}\left\{\frac{[(\Delta^{n+1}X_p)^2]}{[(\Delta^n X_p)^2]}\right\}^{\frac{1}{2}} \quad \text{.....(xxxix).}$$

Thus if the ratio of successive S.D.'s of the differences approaches a limit, that limit will enable us to determine the period, if a single periodic term exists. Again returning to (xxi) we have

$$[(\Delta^n X_p)^2]=\sigma_X{}^2\frac{(2n)!}{n!\,n!}\{2\phi(n,\gamma)-1\}=\sigma_X{}^2\left(2\sin\frac{\gamma}{2}\right)^{2n}$$

or
$$\sigma_X{}^2=\frac{[(\Delta^n X_p)^2]}{\left(2\sin\frac{\gamma}{2}\right)^{2n}} \quad \text{.....(xl),}$$

which enables us after any form of smoothing has given us $\sigma_X{}^2$ to test how far a given periodic term will account for the fluctuations from the secular trend.

Again from
$$\sin\frac{\pi h}{\tau'}=\tfrac{1}{2}\left\{\frac{[(\Delta^{n+1}Y_p)^2]}{[(\Delta^n Y_p)^2]}\right\}^{\frac{1}{2}} \quad \text{.....(xli),}$$

we can determine τ', and thus test whether there is really any common periodic term behind X and Y, as well as whether $\sigma_Y{}^2$ can be to any extent accounted for by such a periodic term.

Lastly suppose that there does exist a common periodic term in X and Y and that

$$X_p=A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\alpha+2\pi(p-\tfrac{1}{2})\frac{h}{\tau}\right\},$$

$$Y_p=A\frac{\tau}{\pi}\sin\frac{\pi h}{\tau}\sin\left\{\beta+2\pi(p-\tfrac{1}{2})\frac{h}{\tau}\right\}.$$

Then the correlation of X_p and Y_{p+s} will be

$$\rho_s=\cos\left(\alpha-\beta-\frac{2\pi sh}{\tau}\right)$$
$$=\cos(\alpha-\beta)\cos\frac{2\pi sh}{\tau}+\sin(\alpha-\beta)\sin\frac{2\pi sh}{\tau}.$$

Now substituting in equation (xviii), it will be clear that the parts of ρ_s and ρ_{-s} with the factor $\sin(\alpha-\beta)$ will cancel, and accordingly

$$\frac{[\Delta^n X_p\Delta^n Y_p]}{\sigma_X\sigma_Y}=\frac{(2n)!}{n!\,n!}\cos(\alpha-\beta)\{2\phi(n,\gamma)-1\}$$
$$=\left(2\sin\frac{\gamma}{2}\right)^{2n}\cos(\alpha-\beta) \quad \text{.....(xlii).}$$

It follows therefore that

$$\frac{[\Delta^{n+1}X_p\Delta^{n+1}Y_p]}{[\Delta^n X_p\Delta^n Y_p]}=\left(2\sin\frac{\gamma}{2}\right)^2$$
$$=\frac{[(\Delta^{n+1}X_p)^2]}{[(\Delta^n X_p)^2]}=\frac{[(\Delta^{n+1}Y_p)^2]}{[(\Delta^n Y_p)^2]} \quad \text{.....(xlii)}\textit{bis},$$

which will serve as a good test of whether the fluctuations can be attributed to a single periodic term, and lastly the difference of phase or "lag" of Y on X will be given by

$$r_{\Delta^n X\,\Delta^n Y}=\cos(\alpha-\beta) \quad \text{.....(xliii)}.$$

Now we know that on "Student's" hypothesis, the ratios of the differences given in (xlii)*bis* tend to the value 4. They would only tend to 4 in the case of a single periodic term if the period were twice the fundamental unit of time, i.e. $\gamma=\pi$, for $\tau=2h$. After the secular terms have been eliminated, the ratios in (xlii)*bis* and the correlation in (xliii) should always be the same in the case of a single periodic term. But in the case of correlated fluctuations, while the correlation in (xliii) would be constant, the ratio of the differences—of course on "Student's" hypothesis—would be $4-\frac{2}{n+1}$ and thus only *tend* to a constant value = 4. On the other hand if ρ_s, ρ_s' and ρ_s'' be not a series of zero correlations (except of course for $s=0$) then the ratios in (xlii)*bis* will tend more slowly to 4, because they depend on quantities like $\frac{n}{n+1}$ being equal to $\frac{n-1}{n}$ practically, as n becomes large.

(6) Having discussed these general points, it is desirable to see what light they throw on the results of our paper of 1914 and how far the criticisms raised by Dr Persons and Mr Yule affect our results.

The work we have undertaken is very laborious and may be considered under three headings:

(*a*) We have checked our arithmetic and calculated the mean products $[\Delta^n x\Delta^n y]$, $[(\Delta^n x)^2]$ and $[(\Delta^n y)^2]$, assuming n sufficiently high to assume that we have got rid of the secular terms. We have then analysed these results.

(*b*) We have smoothed the series by (i) Mr Rhodes' method and by (ii) Dr Sheppard's method, and taking the residuals as X_p, Y_p, have correlated X_p and Y_p, X_p and Y_{p+s}, X_p and X_{p+s}, Y_p and Y_{p+s}.

(*c*) We have applied periodogram analysis to X_p and Y_p to determine if there is really a short period term in the deathrates.

In carrying out this work we have to acknowledge the excellent computing assistance given to us by the members of the Senior Year Statistical Class at University College. They have done a large part of the smoothing, the correlation of the residuals and the periodogram work, computing in pairs and checking their results at each stage. Lengthy as the work has been, it would have consumed even more time of the authors had it not been for this most helpful laboratory-class aid.

We will now state the salient results of that analysis as they apply to our original conclusions and to our critics' statements.

In order to obtain a satisfactory smooth for the period 1859 to 1908, for which we worked in the 1915 paper, we added to our Tables I and II* the more recent results now available. These are given in the following additional tables. The smooth was made by aid of Rhodes' formulae†.

TABLES I α AND II α.

Deathrates in each Year of Life for groups born in the Year of 1st *column.*

Year	Males					Females					Year
	0—1	1—2	2—3	3—4	4—5	0—1	1—2	2—3	3—4	4—5	
1909	120·108	32·285	14·658	8·279	5·751	96·882	30·002	14·608	8·114	5·707	1909
1910	116·245	40·400	13·085	7·841	6·368	94·221	37·836	13·309	7·662	6·508	1910
1911	142·280	31·164	13·373	8·333	7·600	117·366	29·045	12·422	8·311	6·544	1911
1912	105·626	33·862	13·448	10·112	5·523	83·639	30·521	12·898	9·763	5·573	1912
1913	120·145	33·029	17·145	7·671	3·986	96·235	30·464	15·639	7·191	5·959	1913
1914	116·080	40·336	11·779	8·092	10·274	92·755	36·631	10·974	8·086	10·900	1914
1915	122·919	28·584	14·283	14·570	6·774	96·059	26·562	13·291	15·650	6·870	1915
1916	101·995	29·591	21·416	8·013	—	79·894	27·243	21·711	8·109	—	1916
1917	107·607	44·375	13·517	—	—	84·866	43·207	12·944	—	—	1917
1918	107·908	25·040	—	—	—	85·900	22·136	—	—	—	1918
1919	100·003	—	—	—	—	77·603	—	—	—	—	1919

Diagrams I and II (p. 307) are fair representations of the result of smoothing by Rhodes' formulae. The smooth, to judge from the graph, appears to contain no short period terms. The differences of the calculated smoothed values and the observations were now formed for the five columns of the males and are termed X_p, Y_p, Z_p, U_p and V_p respectively for the year p. A search was now made for the short periods with the results given in the following table:

TABLE III.

Amplitude of given Periods Males.

Period	X_p	Y_p	Z_p	U_p	V_p
2·0 years	0·8466	0·6397	0·3212	0·3505	0·3856
2·5 years	1·9095	0·6510	0·1520	0·1705	0·0808
3·0 years	2·9796	0·6485	0·5107	0·4095	0·4232
4·0 years	1·0488	1·2734	0·7089	0·4268	0·4061
5·0 years	3·3139	0·7524	0·7749	0·8328	0·8202
7·0 years	2·0356	1·1308	—	—	—

* *Biometrika*, Vol. x. pp. 491—492.

† *Tracts for Computers*, No. vi. p. 30.

To test whether these results are fairly steady, the results for X_p and Y_p in the case of female babies and children in the second year of life were worked out with the following results:

TABLE IV.

Amplitude of given Periods. Females.

Period	X_p	Y_p
2·0 years	0·2476	0·6387
2·5 years	1·5484	0·4862
3·0 years	2·8312	1·0137
4·0 years	0·6301	1·1310
5·0 years	2·7242	0·9087
7·0 years	1·8783	0·8579

These results are in fair accordance with those of Table III, indicating periods of 3 and 5 years for X_p and 4 years for Y_p.

Now $\frac{1}{2}$(amplitude)^2 will be the contribution of any period to the total standard deviation squared of X_p, Y_p, Z_p, U_p or V_p respectively. These contributions are represented in Table V.

TABLE V.

Contributions to the Total Standard Deviation Squared of the Terms of each Period.

	Males					Females	
	X_p	Y_p	Z_p	U_p	V_p	X_p	Y_p
Total Standard Deviation Squared	40·6622	10·7394	3·3437	3·1893	1·8450	36·0721	10·4871
Periods							
2·0 years	·3584	·2046	·0516	·0614	·0743	·0307	·2040
2·5 years	1·8231	·2119	·0116	·0145	·0033	1·1988	·1182
3·0 years	4·4390	·2103	·1304	·0838	·0895	4·0078	·5138
4·0 years	·5500	·8108	·2513	·0911	·0825	·1985	·6396
5·0 years	5·4910	·2831	·3002	·3468	·3264	3·7106	·4129
7·0 years	2·0718	·6394	—	—	—	1·8018	·3680

In X_p for both male and female the three-year and the five-year periods are the least insignificant but they—if they existed—would not contribute more than $\frac{1}{9}$ and $\frac{1}{7}$ to the total standard deviation squared. Further both these are insignificant periodic terms in male and female Y_p. They cannot therefore possibly be the source of correlation between X_p and Y_p. We can test at once by aid of the table on p. 292 what the relative importance of the terms would be for the three-year period on differencing:

8th difference: $4{\cdot}4390 \times \frac{177147}{27}$ as against $40{\cdot}6622 \times 12870$,

i.e. as 29 to 523

as 1 to 18.

10th difference: $4{\cdot}4390 \times \frac{1594323}{27}$ as against $40{\cdot}6622 \times 184756$,

i.e. as 262 to 7512

as 1 to 29.

In any case there seems no possibility of a three-year period contributing anything of importance to the result or of its swamping the non-periodic but correlated variations. The five-year period would produce even less effect.

It seems to us that Mr Yule should have investigated what real effect his two- or three-year periods would have produced on the result before publishing his criticism. He might further have investigated whether there were any *common* periods in X_p and Y_p, for without such his criticism also falls to the ground. As a matter of fact there are no common periods of any significance at all in X_p and Y_p for either males or females.

We now turn to the correlations as found from the residuals after smoothing; we have for males:

TABLE VI. *Values of the Correlations.*

ρ_1'	= correlation of	X_p and X_{p+1}	$= -{\cdot}3792 \pm {\cdot}0817$
ρ_2'	= „	X_p „ X_{p+2}	$= +{\cdot}1071 \pm {\cdot}0943$
ρ_3'	= „	X_p „ X_{p+3}	$= -{\cdot}1851 \pm {\cdot}0921$
ρ_1''	= „	Y_p „ Y_{p+1}	$= -{\cdot}3352 \pm {\cdot}0847$
ρ_2''	= „	Y_p „ Y_{p+2}	$= +{\cdot}1936 \pm {\cdot}0919$
ρ_3''	= „	Y_p „ Y_{p+3}	$= -{\cdot}3290 \pm {\cdot}0851$
ρ_1'''	= „	Z_p „ Z_{p+1}	$= +{\cdot}0548 \pm {\cdot}0951$
ρ_2'''	= „	Z_p „ Z_{p+2}	$= -{\cdot}2211 \pm {\cdot}0907$
ρ_3'''	= „	Z_p „ Z_{p+3}	$= -{\cdot}2338 \pm {\cdot}0902$
ρ_1^{iv}	= „	U_p „ U_{p+1}	$= +{\cdot}2799 \pm {\cdot}0879$
ρ_2^{iv}	= „	U_p „ U_{p+2}	$= -{\cdot}1565 \pm {\cdot}0932$
ρ_3^{iv}	= „	U_p „ U_{p+3}	$= -{\cdot}3530 \pm {\cdot}0837$
ρ_1^{v}	= „	V_p „ V_{p+1}	$= +{\cdot}3168 \pm {\cdot}0858$
ρ_2^{v}	= „	V_p „ V_{p+2}	$= -{\cdot}3644 \pm {\cdot}0827$
ρ_3^{v}	= „	V_p „ V_{p+3}	$= -{\cdot}5176 \pm {\cdot}0698$.

Now these correlations show that the association of X_p and X_{p+1}, Y_p and Y_{p+1}, Y_p and Y_{p+3}, U_p and U_{p+1}, U_p and U_{p+3}, V_p and V_{p+1}, V_p and V_{p+2}, V_p and V_{p+3} cannot be taken as zero. Thus the criticism of Dr Persons is justified in this case. "Student's" simple hypothesis is not permissible for this material. Now while these correlations as far as X and Y are concerned show an alternating sign, which might be due to a periodic term, the series start with a negative term for death-rates in the first and second year of life, but the correlation becomes positive in the

third, fourth and fifth years of life, being followed by negative correlations; further the magnitudes of these correlations increase for the fourth and fifth years of life. In other words: if there be a two- to three-year period in the deathrates of the first two years of life, there is no correspondingly important period in the third, fourth and fifth years of life.

Supposing only a single substantial period to exist we can determine it from the above correlations. We have to find τ from

$$\rho_s = \cos \frac{2\pi sh}{\tau}.$$

This leads to the following results, the period being in years:

TABLE VII.

Values of Hypothetical Dominant Periods as found from each Deathrate Correlation.

	X	*Y*	*Z*	*U*	*V*
ρ_1	3·206	3·285	4·161	4·882	5·037
ρ_2	2·608	2·562	6·953	7·272	6·466
ρ_3	10·728	9·887	10·433	9·759	8·913

The results for the period as found from ρ_1, ρ_2, ρ_3 are so discordant, that it is very difficult to believe that any one of the deathrates has a single dominant periodicity. The periods roughly suggested for *Z*, *U*, *V* are above the values which would in the least affect our difference results. If there were a two-year to three-year dominant period in *X* or *Y*, it is hard to suppose that ρ_3 in both cases could give so discordant a result, and there is no greater reason for rejecting the ρ_3 value than the ρ_1 or ρ_2. Supposing, however, we do reject it, then if we weight the periods as derived from ρ_1 and ρ_2 according to the probable errors of those correlations we find

Period of $X = 2{\cdot}950$ yrs.,
" " $Y = 2{\cdot}953$ yrs.,

results in agreement with each other and with the period of about three years which we found from direct periodogram analysis.

We see, however, no reason for excluding ρ_3 from such a computation; it has no relatively large probable error, but no result of any value can be obtained by its inclusion. We have already seen that a period of about three years in *X* and *Y* has no amplitude in the least accounting for the observed variations from the secular trend. Further without a lag it would lead to a high *positive* correlation between X_p and Y_p while that correlation is substantial and *negative*. If there be a lag, equation (xliii) shows us that it must be almost a third of a period, or one year, as we might naturally anticipate on the theory of natural selection. But Table IV and the non-accordance of ρ_3 suffice to indicate that the three-year period is of no importance in either X_p or Y_p. Thus Mr Yule's criticism appears to fail when we examine it directly by periodic analysis and indirectly by correlations.

But there are certain general results which may be drawn from these correlations:

(*a*) The geometrical dying out of the correlations is not an hypothesis suitable for application to the present data. Not only do the signs fail in Z, U, V of proper alternation but the magnitudes do not indicate any tendency to die out.

(*b*) In fact the correlations between the mortalities of individuals born in the same year of life at *three* years apart are all negative, and within the limits of their probable errors rise continuously with age. Thus if there be a heavy mortality in children born in year p in their 1st, 2nd, 3rd, 4th and 5th years of life, children born in year $p+3$ will have a light mortality in their 1st, 2nd, 3rd, 4th and 5th years of life. It is difficult to give a reason for such a relationship, especially its increasing intensity with the increasing age of the child.

(*c*) The same relationship holds, in a less marked degree, for children of 3, 4 and 5 years of age born *two* years apart, but it is reversed in the case of children of 1 and 2 years of age, if we can in these cases give any weight to the correlations relatively to their probable errors.

(*d*) Lastly, if we consider children born in successive years (p and $p+1$), then their mortalities in their 1st and 2nd years of life are negatively correlated, and in their 3rd, 4th and 5th years positively correlated. As a matter of fact the correlations of mortalities at the same age of children born in successive years form an algebraic series continually rising $-\cdot3792$, $-\cdot3352$, $+\cdot0548$, $+\cdot2799$, $+\cdot3168$. Of course none of the correlations (V_p and V_{p+3} excepted) between mortalities in the same year of life of children born in different years reach any considerable magnitude, but having regard to their probable errors, they cannot be straightway neglected.

Further they neither fit into geometrically dying out series, nor into anything of the nature of dominant harmonic periods.

A fine piece of work might be done in endeavouring to disentangle these correlations by differentiating the mortality from different causes. It would be very strenuous, but of great service. If it were done, we believe it would be best to try and obtain the crude figures for deaths and populations at risk, and not use any redistributed figures of populations in the 1st, 2nd, 3rd, 4th and 5th years of life.

We next proceed to consider the cross-correlations in order to determine whether we were correct in supposing X_p and Y_p negatively correlated. Still using the smooth by Rhodes' method we find:

TABLE VIII. *Correlations of First and Second Year Mortalities.*

Mortalities Correlated	Boys	Girls
$X_p\ Y_{p-2}$	$-\cdot2077 \pm \cdot0913$	$-\cdot2730 \pm \cdot0883$
$X_p\ Y_{p-1}$	$+\cdot6521 \pm \cdot0548$	$+\cdot5935 \pm \cdot0618$
$X_p\ Y_p$	$-\cdot4584 \pm \cdot0753$	$-\cdot4904 \pm \cdot0725$
$X_p\ Y_{p+1}$	$+\cdot0284 \pm \cdot0953$	$+\cdot0033 \pm \cdot0954$
$X_p\ Y_{p+2}$	$-\cdot2619 \pm \cdot0888$	$-\cdot2838 \pm \cdot0877$

Now these results are exceedingly suggestive and important, and at the same time some are very difficult to interpret. As to the values, those for girls are wholly in keeping with those for boys, the differences being everywhere within the probable errors of the differences. Let us remember that p denotes the year of birth, and the X denotes the excess mortality above the secular trend in the first year of life and Y the excess mortality above the secular trend in the second year of life. Now our results show that the numbers of those born in the same year who die in the first and the second year of their lives are *negatively* correlated, and that this correlation is substantial, somewhat under $-\cdot5$. Thus the conclusion reached in the former paper by the present writers is confirmed by an entirely different form of investigation, but the values now obtained are 33 °/₀ less than those obtained by the variate-difference method. This reduction undoubtedly flows from the neglect of correlations such as those of $X_p X_{p+s}$, $X_p Y_{p+s}$, $Y_p Y_{p+s}$ on "Student's" hypotheses. That neglect was not in our case justifiable, for it gave an exaggerated value to the influence of natural selection. That influence however still remains substantial.

Turning to the correlation of X_p and Y_{p-1}, i.e. the mortalities of children born in successive years, dying in the same year, we see that it is high and positive; it is probably due to diseases peculiar to certain years and attacking not only infants, but also children in the second year of life. Our direct value for the correlation of X_p and Y_p shows that the high positive correlation of X_p and Y_{p-1} is *not* the source of the negative correlation between X_p and Y_p as was suggested by one of our critics.

The correlation $X_p Y_{p+1}$ of deaths of infants born in year p with those of two-year-olds born in the following year is zero sensibly, as we should naturally anticipate. But we should also have anticipated that there would be no correlation ($X_p Y_{p+2}$) between the deaths of infants born in year p and that of two-year-olds born two years later and therefore dying three years later! The correlations are not very large, but they appear in both sexes, and it is difficult to consider them zero having regard to their probable error. Such correlation might be accounted for by a periodic term common to X and Y with a suitable difference of phase, but to judge by Table VI there seems no common period of sufficient importance to justify the suggestion.

Lastly the first correlation of Table VIII is again one difficult of interpretation. The correlation of X_p and Y_{p-2} might possibly be considered zero for boys, but the value for girls suggests that it is really significant. Why should the dying of children in their second year in one year, be related to those dying in their first year in the following year? Had the correlation been *positive*, we might have considered epidemics lasting from one year into a second, but the correlation is *negative*.

The only explanation we have been able to think of will we fear be thought by many inadequate. It is that Y_{p-2}, X_p and Y_{p+2} are children born two years apart, and the *average* interval between children in the same family is somewhat over two years. Accordingly the groups mentioned above correspond to a considerable extent

to children of the *same* families. It is possible that when the previously born child has died greater care is taken of the next infant, and further that when a mother has a child alive between 1 and 2, her attention is not so wholly devoted to the new-born child and it may be less likely to survive. It may well be that the falling birthrate is really organically related to a falling infant deathrate, because it connotes fewer extremely young children to tend simultaneously*.

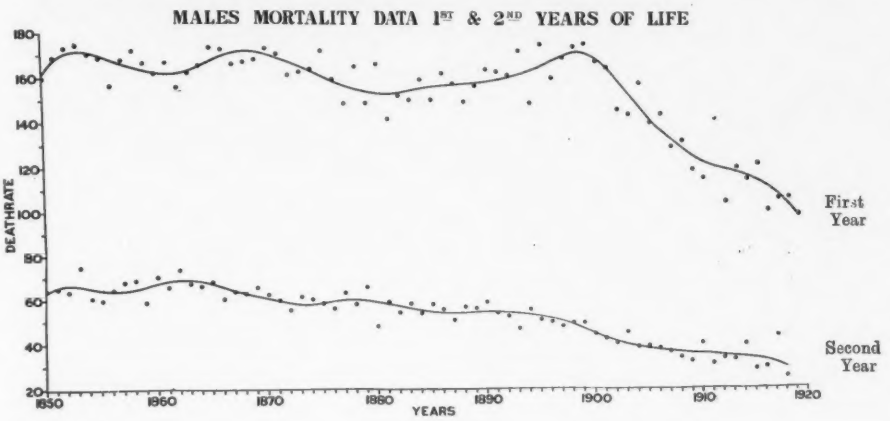


Diagram I.

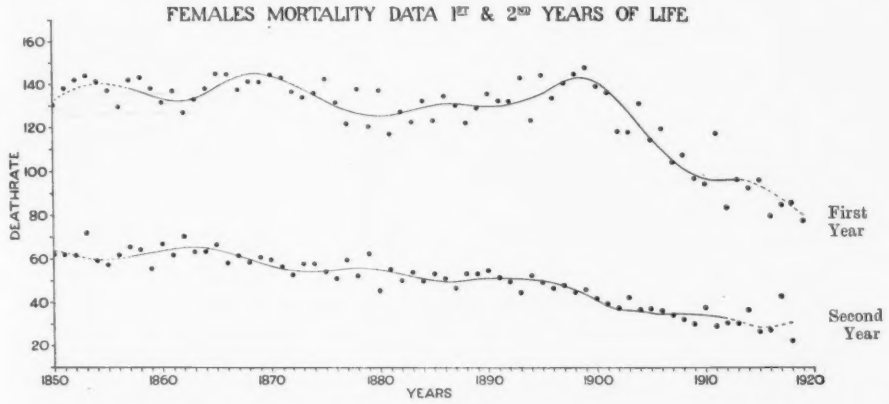


Diagram II.

We shall now show that our values for the correlation of X_p and Y_p are not peculiar to the particular method of smoothing adopted. The smooths of X_p and Y_p are shown in Diagrams I and II for boys and girls respectively. It will be seen

* We are well aware that the environmentalists attribute it largely to the spread of infant welcome and mother welfare centres, but we doubt if they have investigated the correlation between infant deathrate in towns with and without these centres.

on careful examination of these diagrams that the general rule is *reversal* of sign in X_p and Y_p. In Diagram I in 69 years there are 43 reversals of sign, in Diagram II there are 41 reversals, or roughly $\frac{3}{5}$ of the total. Thus graphical inspection shows a very considerable degree of negative correlation between deaths in the first and second years of life of the same group of individuals. But would another system of smoothing give the same general trend and an equal correlation? This was tested by an entirely different process of smoothing, namely that of Sheppard.

The result was*:

Sheppard's process: $r_{X_p Y_p} = -\cdot 4629$ (Boys)
Rhodes' process: $r_{X_p Y_p} = -\cdot 4584$ (Boys)
" " $r_{X_p Y_p} = -\cdot 4904$ (Girls)

We think it safe to say that there really does exist a substantial *negative* correlation between deaths of the same group in the first and second years of life. It is not as great as we found it in the previous paper using hypotheses, which, we admit, ought to have been tested; but it is quite adequate to indicate that natural selection is really at work.

The value of the $X_p Y_p$ correlations was exaggerated by adopting "Student's" hypotheses; the negative sign, however, was not, as suggested by Mr R. A. Fisher (*Journal of R. Statistical Society*, Vol. XXXIV. p. 534), due to the neglect of the correlations of children of different ages dying in the same year. Those correlations are very considerable and undoubtedly aid in modifying the value of $r_{X_p Y_p}$ as found by "Student's" method. Thus we have

$r_{X_p Y_{p-1}}$: for boys ·6521 and for girls ·5935
$r_{Y_p Z_{p-1}}$: " ·6851
$r_{Z_p U_{p-1}}$: " ·9050
$r_{U_p V_{p-1}}$: " ·7481

from which it appears that the liability of children to the same seasonal diseases is greatest in the 3rd and 4th years of life.

The increased negative value to the numerator of the expression (xxiv) due to the introduction of further negative terms (including of course $r_{X_p Y_{p-1}}$) is, however, corrected by increased *positive* terms in the expressions for $[(\Delta^n X_p)^2]$ and $[(\Delta^n Y_p)^2]$ of the denominator.

Assuming the difference order n to be 8, we calculated from the X_p, X_{p+s}, Y_p, Y_{p+s} and X_p, Y_{p+s} correlations, as far as we had worked them out, the values of

$$\phi(n, \rho_0, \rho), \text{ and } \phi(n, \rho'), \phi(n, \rho''),$$

and found, using seven terms in each of the latter and five in the former, that:

$$r_{\delta_s x_p \delta_s y_p} = -\cdot 68,$$

* Fitting single sixth order parabolas to X_p and Y_p was tried for 67 years, but they gave very inadequate smooths; it wants more than a single sixth order parabola to adequately describe the trends.

whereas "Student's" process gave the value $-\cdot 69$; the $-\cdot 01$ difference could very easily be accounted for by the small number of X_p and Y_{p+s} correlations that we were able to use.

This correspondence in value gives us some confidence in the proposal to deduce the $r_{X_p Y_{p+s}}$ and $r_{X_p X_{p+s}}$, $r_{X_p Y_{p+s}}$ correlations—when we are certain that the correlations die out rapidly with s—from the extended Variate Difference Equations given in this paper. If this method is to give satisfactory results, however, we think it will be desirable to work with the same order of differences, and reach enough equations by correlating X_p with Y_{p+s}, X_p with X_{p+s} and Y_p with Y_{p+s}, i.e. not corresponding differences. We reserve, however, this matter for a fuller consideration later, probably on other material. It may turn out that the process is more lengthy than subtracting the trend by an adequate smooth, which at least enables us to correlate only the values we need.

We have published Mr Henderson's Table for geometrical decadence functions, because although it is of no service for these mortality data, we think it may well be so for other material. The law of geometrical decadence seems a reasonable one for many variates. We should like to try it on the growth data, say at weekly or monthly intervals for a series of the same years of age, in the case of brothers, to ascertain whether "spurts" and "checks" to growth are similar in two members of the same family. And we should welcome data of this kind that any reader of this paper may have in his power to lend us.

Our general conclusions so far must be

(i) that Mr Yule's criticisms as to periodic terms are not valid and that this could have been seen had the material been analysed at greater length;

(ii) that Dr Persons' criticism is valid, although his attempts to get rid of secular trends are from our standpoint wholly inadequate, and further his criticism was hypothetical. He suggested, but he did not show, that X_p and X_{p+s} or X_p and Y_{p+s} were in any of our cases really correlated.

That "Student's" hypotheses did not truly apply to our mortality data has, however, been shown in this paper. We do not think this signifies that the Variate Difference Method, either in its old, or in a more generalised form, will be of no service. It denotes only that it was not suited to the data to which we applied it. Because a non-justifiable method was adopted it does not follow that the conclusion we reached was erroneous. The present paper, we think, shows that mortality in the first year of life is negatively correlated with that in the second; the correlation is substantial but only about 70 °/₀ of what we deduced for it.

APPENDIX.

Table of Functions for Geometrical Decadence (J. HENDERSON).

$n=$	0	1	2	3	4	5	6	7	8	9	10
$\epsilon=-1\cdot0$	1·0	2·0	2·666667	3·200000	3·657144	4·063492	4·432900	4·773892	5·092152	5·391690	5·675460
	4·0	**4·0**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**	**4·000000**
$-0\cdot9$	1·0	1·9	2·470000	2·908900	3·273375	3·588544	2·868073	4·120351	4·350953	4·563806	4·761794
	3·8	**3·9**	**3·925641**	**3·938538**	**3·946617**	**3·952281**	**3·956534**	**3·959875**	**3·962591**	**3·964852**	**3·966778**
$-0\cdot8$	1·0	1·8	2·280000	2·635200	2·920732	3·160823	3·368545	3·551856	3·716004	3·864657	4·000490
	3·6	**3·8**	**3·852632**	**3·879236**	**3·895928**	**3·907632**	**3·916412**	**3·923305**	**3·928902**	**3·933560**	**3·937514**
$-0\cdot7$	1·0	1·7	2·096667	2·378300	2·597260	2·776223	2·927253	3·057601	3·171987	3·273671	3·365004
	3·4	**3·7**	**3·781080**	**3·822230**	**3·848056**	**3·866138**	**3·879680**	**3·890289**	**3·898882**	**3·906017**	**3·912058**
$-0\cdot6$	1·0	1·6	1·920000	2·137600	2·301074	2·430903	2·537778	2·628011	2·705649	2·773438	2·833334
	3·2	**3·6**	**3·711111**	**3·767664**	**3·803116**	**3·827873**	**3·846351**	**3·860784**	**3·872429**	**3·882066**	**3·890197**
$-0\cdot5$	1·0	1·5	1·750000	1·912500	2·030357	2·121280	2·194298	2·254630	2·305556	2·349265	2·387288
	3·0	**3·5**	**3·642857**	**3·715686**	**3·761214**	**3·792880**	**3·816410**	**3·834702**	**3·849398**	**3·861503**	**3·871673**
$-0\cdot4$	1·0	1·4	1·586667	1·702400	1·783360	1·844018	1·891551	1·930008	1·961877	1·988787	2·011857
	2·8	**3·4**	**3·576471**	**3·666447**	**3·722448**	**3·761182**	**3·789801**	**3·811921**	**3·829596**	**3·844080**	**3·856190**
$-0\cdot3$	1·0	1·3	1·430000	1·506700	1·558403	1·596019	1·624793	1·647603	1·666177	1·681623	1·694687
	2·6	**3·3**	**3·512121**	**3·620104**	**3·686895**	**3·732772**	**3·766430**	**3·792275**	**3·812799**	**3·829521**	**3·843425**
$-0\cdot2$	1·0	1·2	1·280000	1·324800	1·353874	1·374415	1·389761	1·401690	1·411245	1·419079	1·425624
	2·4	**3·2**	**3·450000**	**3·576811**	**3·654619**	**3·707607**	**3·746168**	**3·775563**	**3·798749**	**3·817526**	**3·833051**
$-0\cdot1$	1·0	1·1	1·136667	1·156100	1·168231	1·176556	1·182634	1·187272	1·190930	1·193891	1·196337
	2·2	**3·1**	**3·390322**	**3·536726**	**3·625654**	**3·685609**	**3·728852**	**3·761554**	**3·787171**	**3·807785**	**3·824744**
0·0	1·0	1·0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	2·0	**3·0**	**3·333333**	**3·500000**	**3·600000**	**3·666667**	**3·714286**	**3·750000**	**3·777778**	**3·800000**	**3·818182**
0·1	1·0	0·9	·870000	·855900	·847774	·842508	·838824	·836104	·834015	·832361	·831019
	1·8	**2·9**	**3·279310**	**3·466771**	**3·577638**	**3·650634**	**3·702242**	**3·740631**	**3·770286**	**3·793873**	**3·813080**
0·2	1·0	0·8	·746667	·723200	·710217	·702029	·696411	·692323	·689218	·686780	·684817
	1·6	**2·8**	**3·228571**	**3·437167**	**3·558496**	**3·637324**	**3·692483**	**3·733182**	**3·764415**	**3·789139**	**3·809176**
0·3	1·0	0·7	·630000	·601300	·586060	·576695	·570381	·565845	·562432	·559772	·557642
	1·4	**2·7**	**3·181481**	**3·411292**	**3·542473**	**3·626522**	**3·684748**	**3·727381**	**3·759911**	**3·785541**	**3·806240**
0·4	1·0	0·6	·520000	·489600	·474103	·464808	·458638	·454253	·450979	·448443	·446421
	1·2	**2·6**	**3·138461**	**3·389217**	**3·529420**	**3·617995**	**3·678774**	**3·722972**	**3·756534**	**3·782866**	**3·804076**
0·5	1·0	0·5	·416667	·387500	·373214	·364832	·359341	·355473	·352603	·350391	·348633
	1·0	**2·5**	**3·100000**	**3·370965**	**3·519148**	**3·611481**	**3·674305**	**3·719723**	**3·754079**	**3·780934**	**3·802521**
0·6	1·0	0·4	·320000	·294400	·282331	·275384	·270881	·267731	·265405	·263618	·262202
	0·8	**2·4**	**3·066667**	**3·356517**	**3·511419**	**3·606711**	**3·671094**	**3·717421**	**3·752342**	**3·779589**	**3·801440**
0·7	1·0	0·3	·230000	·209700	·200460	·195224	·191856	·189512	·187788	·186466	·185420
	0·6	**2·3**	**3·039130**	**3·345780**	**3·505968**	**3·603410**	**3·668906**	**3·715886**	**3·751183**	**3·778684**	**3·800726**
0·8	1·0	0·2	·146667	·132800	·126674	·123243	·121047	·119524	·118406	1·117550	·116874
	0·4	**2·2**	**3·018181**	**3·338547**	**3·502493**	**3·601332**	**3·667553**	**3·714923**	**3·750467**	**3·778147**	**3·800290**
0·9	1·0	0·1	·070000	·063100	·060117	·058459	·057399	·056667	·056128	·055716	·055393
	0·2	**2·1**	**3·004762**	**3·334540**	**3·500553**	**3·600320**	**3·666918**	**3·714351**	**3·750048**	**3·777867**	**3·800053**
1·0	1·0	0·0	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000	·000000
	0·0	[2·0]	[3·000000]	[3·333333]	[3·500000]	[3·600000]	[3·666667]	[3·714286]	[3·750000]	[3·777778]	[3·800000]

The above table gives $2\phi(n,\ \epsilon)-1$ in ordinary type and $\left(4-\frac{2}{n+1}\right)\left\{\frac{2\phi(n+1,\epsilon)-1}{2\phi(n,\epsilon)-1}\right\}$ in heavy type for values of n from 0 to 10 and of ϵ from $-1\cdot0$ to $+1\cdot0$ at *intervals* of 0·1.

$$\left[\phi(n,\epsilon)=1-\frac{n}{n+1}\epsilon+\frac{n(n-1)}{(n+1)(n+2)}\epsilon^2-\ldots(-1)^n\frac{n!}{(n+1)(n+2)\ldots(2n)}\epsilon^n.\right]$$

FACIAL SPASM INHERITED THROUGH FOUR GENERATIONS.

By PERCY STOCKS, M.D.

A CASE of facial spasm which was of considerable interest from the hereditary standpoint recently came to the notice of the Anthropometric Laboratory, University College, London, and is here recorded. As there has been much confusion in the classification and nomenclature of affections of this type, a few preliminary notes on the modern views regarding facial spasm are necessary. The confusion which has in the past existed between so-called "tics" and "spasms," both of which are evidenced by twitching movements of the face muscles, has been chiefly responsible for the large number of names used to describe these affections, such as "facial tic," "spasmodic tic," "convulsive tic," "habit spasm," "mimic spasm," "tic non-douloureux," and others. Confusion with choreic movements has also led to such names as "habit chorea."

The distinction between "tic" and "spasm" has been made clear in recent years by the researches of Charcot and other French workers, leading up to the excellent work of Meige and Feindel (2). *Spasm* is now defined by modern writers as a motor reaction which results from irritation at some point in a reflex spinal or bulbo-spinal arc, and neither depends on consciousness nor is under the influence of the will. *Tic*, on the other hand, is essentially a psychical process in which a co-ordinated muscular movement is repeated intermittently for no useful purpose, and in which consciousness is essential and the exercise of the will plays an important part.

Facial tic (also called habit spasm, convulsive tic, &c.) is therefore characterised by frequent or intermittent clonic contractions of a co-ordinated group of facial muscles (not strictly limited to those supplied by any one nerve), occurring commonly in children or young persons with some neuropathic tendency or heredity. It is never entirely beyond the control of the will, never occurs during sleep, and each paroxysm is usually preceded by an impulse, and followed by a certain feeling of satisfaction. These tics are of great variety, but they need not be enumerated here. Heredity is not infrequently manifest in this affection, but owing to its very nature, it is difficult to eliminate the element of imitation, which may be the starting point of the habit in a child of unstable nervous temperament, whose parent or brother or sister is the subject of a tic. Several cases of apparent direct heredity have been recorded; thus Sir W. Gowers (1) mentions the case of a father and two children with identical tics; and Piedagnel (2) a case where mother and daughter had similar tics. Tissié (2) describes ocular tic occurring in mother and two sons, and Letulle (2) a similar tic in father and

two sons. Strauch (3) mentions two cases; in one a boy, his mother, two maternal aunts and one maternal uncle all had tics in their youth; in the other a girl, both her parents and one uncle were affected. Other cases, where two or more children in a family developed identical tics, have been mentioned by Gintrac (2) (two brothers), Blache (4) (3 children), Delasiauve (2) (brother and sister), Meige and Feindel (2) (two sisters) and Flatau (5) (two sisters). Similar tics in two brothers are also recorded by Rudler and Chomel (11). Whilst direct heredity is not very often traced, a neuropathic heredity is frequent; thus it is common for example to find that the mother of a ticqueur is hysterical, a brother epileptic and a grandparent insane. Flatau (5) describes two such cases; in one the mother and her sister had tics and the son was mentally abnormal; in the other the mother of a ticqueur had impulsive insanity. It is also common to find mental instability in one parent combined with intellectual brilliance in the other.

Facial Spasm is characterised by spasms, usually clonic but occasionally tonic, occurring in muscles supplied by the facial nerve. These are most often, but not always, unilateral, and may affect all the muscles or only those supplied by one or more branches of the nerve. The affection may be secondary to paralysis; or result from direct irritation of the seventh (facial) nerve; or from reflex irritation of the fifth (trigeminal) nerve by a carious tooth, injury, ocular defect, ear disease or tumour; or from injury to the facial area in the cortex; or from a tumour pressing on the pons. In many cases (labelled "idiopathic") no definite cause can be assigned, but shock, nervous emotion, exposure to cold, prolonged migraine or neuralgia, pregnancy, and the climacteric have all been mentioned as precursors, whilst a few have been attributed to heredity. The idiopathic form is more frequent in females than males and usually occurs between 45 and 60 years of age, and rarely before 30.

Pathological evidence shows that the facial nucleus in the pons is the starting point of the motor discharges resulting in spasm, and that these are a pure reflex in which the cortex takes no part (6).

Paroxysms may occur at intervals of several hours, or recur every minute or two; they generally consist of clonic muscular contractions which begin more or less slowly and become quicker and more pronounced until they may be so rapid as to have a quivering character; the paroxysm then slows down and finally ceases. Emotional excitement, cold, or a bright light often make the spasms more pronounced. In the early stages a single muscle, or part of a muscle (7), is usually involved, but the spasm usually tends to become more generalised.

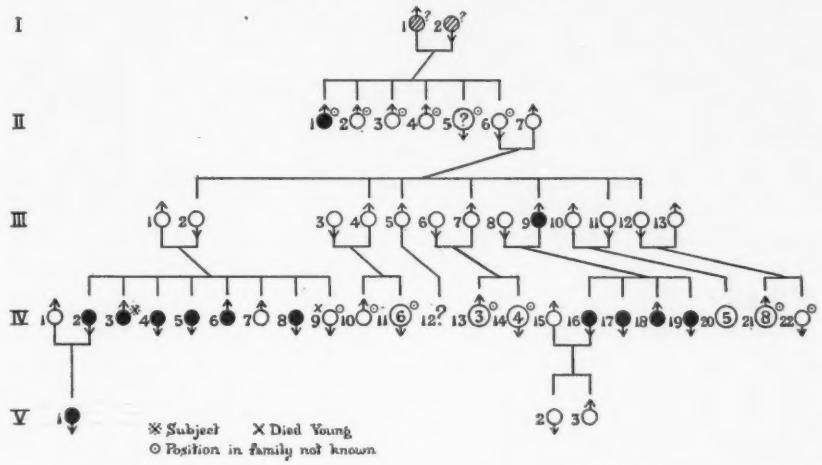
Gowers (1) states that direct heredity is very rare, but mentions a case where mother and daughter were both affected in late life. Rosenthal (8) speaks of a family where mother, son, daughter and two maternal relatives were affected by more or less extensive facial spasm, but this is not described in detail. Mayer (9) describes the case of a man with infra-orbital neuralgia giving place after the lapse of five years to spasm of the left eyelid and finally of the whole side of the face; his mother had been similarly affected, but in her case the spasm

preceded the neuralgia. Probably the last were really cases of tic douloureux or painful facial spasm.

The hereditary cases described by Piedagnel, Blache, Delasiauve and Gintrac, which are sometimes alluded to under the heading of facial spasm, seem to have been most probably true cases of tic, and have been referred to above as such. Gowers states that an inherited neurotic tendency to insanity or epilepsy can sometimes be traced.

The case we shall now describe is unusual in several respects, and appears to demonstrate the hereditary transmission of the affection more completely than any case yet described.

Pedigree of Familial Facial Spasm.



The subject who came under our notice was a male of 18 years of age, of Polish parentage on both sides, and the second child in a family of 8. His general health had been excellent; he weighed 87 kilograms, and his stature was 178 cm. His respiratory measurements were high and his circulatory system showed nothing abnormal. In a series of mental speed tests he was uniformly brilliant and in tests of muscular power was also in the first grade, whilst both visual and auditory acuity were much above normal. In muscular precision and discrimination of weights he made average scores, but was distinctly subnormal in tests of sensory discrimination, and failed completely in a balancing test designed to measure steadiness of hand and arm. These results confirmed an impression of a good physique and unusually quick intellect combined with a somewhat restless and highly strung temperament.

Throughout the examination there was noticeable a very rapid clonic spasm localised to the levator menti muscle between the chin and lower lip. This spasm

was often so rapid as to become a quivering movement or tremor, and ceased for short intervals, becoming more pronounced during the performance of psychological tests requiring sustained effort. It was never observed to spread to any other muscle, and according to his own account no other parts of the face or body were ever affected. The whole muscle was apparently involved and there was no sign of any predominant lateral distribution of the spasm.

On inquiring as to heredity, he asserted definitely that an identical affection was present in one of his brothers and four sisters, in a daughter of one of those sisters, in a maternal uncle and four children of the latter (one male and three female), and in a brother of his maternal grandmother. His parents and grandparents were themselves free. From further details the accompanying pedigree was constructed, from which it will be noticed that:

(i) The affection was manifested in 13 individuals in 4 generations. About his great grandparents the subject could give no information.

(ii) Of these 13, 8 were females and 5 males, a proportion agreeing closely with Gowers' statement that the disease occurs predominantly in females in the proportion of about two to one.

(iii) Transmission occurs both through affected males and females, and there is an apparent tendency for its prevalence to increase with succeeding generations; at any rate the proportions of total children affected in generations II to V were successively $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{4}$, $\frac{1}{3}$. Where an affected male transmitted the disease, all his children were affected.

(iv) Transmission through an unaffected female occurs twice in succession, an unaffected mother and her daughter carrying the disease to six grandchildren. This of course occurs in many hereditary anomalies, but in the case of nervous affections has rarely been demonstrated. Thus Huntingdon's Chorea is stated by Huntingdon never to skip a generation (10), and I am not aware that it has been demonstrated in Paramyoclonus multiplex or other hereditary nervous affections.

The occurrence of insanity, epilepsy or any other neuropathic tendency in this family was denied. Another unusual feature is the appearance of the affection at an early age in some members of the family. A few instances have, however, been met with where the onset was noticed as early as 20. This case presented none of the features characteristic of a tic or habit spasm, and there appears little doubt as to the diagnosis. Gowers states that where the spasm is localised to one muscle, the orbicularis palpebrarum or zygomatici are most commonly affected, but that the depressor anguli oris or levator menti are sometimes implicated. According to the information supplied, the affection remained entirely confined to the levator menti region in all the affected members of this family.

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A BIOMETRIC STUDY OF THE INTER-RELATIONS OF "VITAL CAPACITY," STATURE, STEM LENGTH AND WEIGHT IN A SAMPLE OF HEALTHY MALE ADULTS.

By LUCY D. CRIPPS, MAJOR GREENWOOD and ETHEL M. NEWBOLD.

[*Being a paper read before the Society of Biometricians and Mathematical Statisticians, October* 23, 1922.]

Introduction.

In any study of physical fitness or efficiency the first difficulty to be encountered is that of the definition of "Fitness." A more or less satisfactory definition having been framed, one seeks for a variable magnitude highly correlated with the complex defined as "fitness," so highly that it may itself be deemed a measure of "fitness." This point having been reached, we have to ascertain whether "fitness" so measured is correlated with other measurable physical or mental attributes to such an extent that, the latter measurements of an individual being assigned, his or her "fitness" can be estimated within reasonably narrow limits of accuracy. Eighty years ago, Dr John Hutchinson* concluded that "vital capacity," that is, the volume of the maximum expiration following a maximum inspiration, was a valuable measure of "fitness" and that its variations were highly correlated with certain bodily dimensions. More recent investigators have endorsed Hutchinson's contentions in principle, although they have modified the form of the relation between "vital capacity" and other variables, and made a different choice of the latter.

This memoir is wholly concerned with the second part of Hutchinson's and his successors' problem. We shall not discuss the question whether "vital capacity" be or be not an adequate measure of "fitness," but confine ourselves exclusively to the simple biometric question—with what success can "vital capacity" be expressed as a function of the variables, stature, stem length (or sitting height), and body weight when either linear equations or simple exponential functions of the independent variables are employed.

Previous Investigations.

We shall not attempt to give a complete historical study of the work stimulated by the publication of Hutchinson's results, but an abstract of the original work and a short reference to subsequent investigations will probably interest the reader.

* "On the Capacity of the Lungs and on the Respiratory Functions with a view of establishing a precise and easy method of detecting Disease by the Spirometer." *Transactions of the Royal Medico-Chirurgical Society of London*, Vol. xxix.

TABLE I.

Table of Mean Vital Capacity of 15 *Different Classes considered as Healthy.* (HUTCHINSON.)

	ft. in. 0 0—5 0		ft. in. 5 0—5 1		ft. in. 5 1—5 2		ft. in. 5 2—5 3		ft. in. 5 3—5 4		ft. in. 5 4—5 5		ft. in. 5 5—5 6	
	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases
Seamen	2474	5	3376	1	3146	7	3589	1	3572	10	3490	9	3556	15
Fire Brigade ...	—	—	—	—	—	—	—	—	3441	1	3408	2	3572	20
Police (Met.) ...	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Police (Thames)	2589	1	—	—	—	—	—	—	3064	6	3376	9	3736	9
Paupers	2474	7	2720	3	2655	10	2950	10	2851	21	3130	20	3097	19
Mixed Classes ...	1311	1	3032	1	2655	5	2966	5	3032	17	3130	16	3146	20
Grenadier Guards	—	—	—	—	—	—	2753	1	—	—	3572	1	3261	2
Compositors ...	—	—	—	—	2884	3	2704	2	3212	5	3081	6	3408	7
Pressmen ...	—	—	2491	1	—	—	—	—	—	—	3490	2	3326	8
Draymen ...	—	—	—	—	—	—	—	—	—	—	—	—	3146	1
Gentlemen ...	—	—	—	—	2376	1	2638	1	2556	7	2900	9	3097	14
Pugilists, etc. ...	—	—	—	—	3310	1	3572	2	3572	1	3458	4	3556	3
Horse Guards ...	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Mean of 1st series (unweighted)	2212	14	2900	6	2835	27	3015	22	3163	68	3408	78	3343	118
Chatham recruits	—	—	2737	1	2966	1	—	—	3097	1	—	—	3818	19
Woolwich marines	—	—	—	—	—	—	—	—	—	—	3540	3	3654	7
Miscellaneous ...	—	—	2950	1	—	—	3179	4	3245	4	2950	4	3212	10
Total mean under each height (unweighted)	2212	14	2868	8	2900	28	3097	26	3163	73	3294	85	3507	154
Ditto corrected for weighting	2399	—	2843	—	2827	—	3022	—	3049	—	3208	—	3406	—

	ft. in. 5 6—5 7		ft. in. 5 7—5 8		ft. in. 5 8—5 9		ft. in. 5 9—5 10		ft. in. 5 10—5 11		ft. in. 5 11—6 0		ft. in. 6 0—6 1		Total	Weighted Mean
	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	c.c.	cases	cases	c.c.
Seamen	3703	14	3753	15	3916	11	4228	18	4474	12	4424	6	4031	2	126	3790
Fire Brigade ...	3523	17	3785	26	3785	20	3884	3	4261	1	4080	2	—	—	92	3693
Police (Met.) ...	3834	4	3736	33	3703	46	4064	22	3834	13	4293	12	4605	11	141	3903
Police (Thames)	3638	15	4031	17	4097	10	3933	5	4211	3	—	—	—	—	75	3751
Paupers	3441	10	3064	9	3261	10	4293	1	3933	3	—	—	—	—	123	3037
Mixed Classes ...	3441	20	3638	28	3900	16	4031	14	3900	7	4408	9	—	—	159	3481
Grenadier Guards	—	—	3736	7	3818	22	3933	16	3802	11	4146	9	4228	14	83	3907
Compositors ...	3720	5	3523	8	3507	6	3785	3	—	—	4146	1	—	—	46	3387
Pressmen ...	3343	3	3654	7	4015	1	3916	4	4048	2	—	—	—	—	28	3552
Draymen ...	3949	1	3572	3	3654	4	4015	1	4277	6	4064	4	—	—	20	3918
Gentlemen ...	3408	10	3408	18	3408	16	3867	8	4162	12	4097	5	4293	5	106	3445
Pugilists, etc. ...	4375	3	3376	1	3982	2	4474	3	4457	5	4064	2	—	—	27	3957
Horse Guards ...	—	—	—	—	—	—	—	—	—	—	4179	30	4506	26	56	4331
Mean of 1st series (unweighted)	3671	102	3605	172	3753	164	4031	98	4162	75	4179	80	4261	58	—	—
Chatham recruits	3900	67	4048	38	4113	22	4359	16	3867	2	4277	5	4654	3	175	3999
Woolwich marines	3818	99	3851	192	3933	130	4031	75	4097	39	4310	18	4523	9	572	3925
Miscellaneous ...	3638	18	3490	9	3769	13	3703	12	—	—	4244	7	4687	6	88	3645
Total mean under each height (unweighted)	3753	286	3736	411	3884	329	4031	201	4048	116	4244	110	4523	76	1917	—
Ditto corrected for weighting	3739	—	3786	—	3836	—	4042	—	4094	—	4239	—	4465	—	—	—

Hutchinson took as his chief criterion vital capacity, i.e. the volume of the greatest expiration following the deepest inspiration. This differs from the total lung volume by the amount of the residual air, i.e. the quantity of air which always remains in the lungs and over which we have no control. Having chosen his measure, he next proceeded to investigate its normal value in healthy people as a standard of measurement. In examining the work of his predecessors on vital capacity, Hutchinson found various statements as to its mean value (varying from 1637 c.c. to 4916 c.c.), but few direct experiments and these on very small numbers. Hutchinson's great advance on previous work was that, besides using direct experiment on large numbers, he brought in the idea of proportion and tried to establish normal relations between vital capacity and other physical measurements. He says:

"One of the fundamental rules in architecture is proportion, the relation that the whole fabric has to its constituent parts and which each part has to the complete idea of the whole, for in buildings that are perfect in their kind, from any particular part an architect may form a tolerable judgment of the whole: just in like manner the physiologist from a portion of the viscera, say an organ, should be able to form a tolerable judgment of the whole man from whom it was taken."

The other observations that Hutchinson took besides vital capacity were:

Power of Expiratory Muscles.
Power of Inspiratory Muscles.
Circumference of the Chest over the nipples.
Height.
Weight.
Pulse (sitting).
Respirations per minute (sitting).
Age.
Temperature of air breathed into the spirometer.
Remarks on the occupation and general appearance.

TABLE II.

Height and Vital Capacity in Arithmetical Progression. (HUTCHINSON.)

Height ft. in.	Series from 1012 cases cubic inches	Series from 1917 cases cubic inches	Series from Arithmetical Progression cubic inches	Arithmetical Progression cubic cms.
5 1	175·0	176·0	174·0	2851
5 3	188·5	191·0	190·0	3113
5 5	206·0	207·0	206·0	3376
5 7	222·0	228·0	222·0	3638
5 9	237·5	241·0	238·0	3900
5 11	254·5	258·0	254·0	4162
Means of all heights	214·0	217·0	214·0	3507

These records were made on 2130 persons of the following classes:

Sailors (Merchant Service)	121
Fire Brigade of London	82
Metropolitan Police	144
Thames Police	76
Paupers	129
Mixed Class (artisans)	370
1st Battalion Grenadier Guards ...	87
Royal Horse Guards (Blues)	59
Chatham Recruits	185
Woolwich Marines	573
Pugilists and Wrestlers	24
Giants and Dwarfs	4
Printers (Pressmen 30, Compositors 43)	73
Draymen	20
Girls	26
Gentlemen	97
Diseased cases	60
Total	2130

The mean values of the vital capacities that he observed for these at different heights are given in Table I (cubic inches are converted into cubic centimetres for comparison with others), which is typical of the way he presents his data. The means for each height (at the bottom of the table), as Dreyer has pointed out, are wrong as Hutchinson had not weighted his groups. (To the last column of means has been added the corrected means for each height.)

To get the relation between vital capacity and height he prepared two similar tables grouping his classes of people together, calculated the mean of vital capacity every two inches of height, and so obtained Table II. The last column in this is the nearest Arithmetical Progression obtained apparently by inspection. His result was:

"For every inch of height from 5 feet to 6 feet eight additional cubic inches of air at 60° are given out by a forced expiration."

He treated his weight table in the same manner, sub-dividing for height at the same time, but found that the effect of weight was not so definite as that of height. He concluded that weight did not affect vital capacity until it was in excess of what it should be for a given height; but the difficulty arises, that there exists no standard of weight for given heights. He found that the relation between vital capacity and weight was not linear throughout. Vital capacity ascended up to 160 lbs. and then remained nearly stationary up to 200 lbs. More exactly, vital capacity increased with weight in the ratio of 1 cubic inch to 1 lb. from 105 to 155 lbs. and from 155 to 200 lbs. there is a loss of 39·5 cubic inches. This

refers to heights of 5 feet 6 inches and he modified it for other heights. Dreyer has re-calculated this, weighting the various groups properly, and finds that when this is done vital capacity ascends with weight.

Hutchinson next tried to find a standard relation between weight and height, he added other observations and so brought his males up to 3000 of ages from 15 to 40 years. Table III shews his scale of weights for different heights. He says that instead of varying as the cube of the height as it should do if the body were symmetrical, weight does in fact vary as the 2·75th power of the height.

TABLE III.

Mean Weight (including clothes) in Relation to Height of 3000 *Males. Ages* 15 *to* 40 *years.* (HUTCHINSON.)

Heights	Number of Cases	Mean Weight in lbs.	Mean Weight in kilograms
ft. in. ft. in.			
4 6 to 5 0	26	92·26	41·85
5 0 5 1	17	115·52	52·40
5 1 5 2	36	124·33	56·40
5 2 5 3	43	127·86	58·00
5 3 5 4	88	138·01	62·60
5 4 5 5	126	139·17	63·13
5 5 5 6	214	144·93	65·74
5 6 5 7	316	144·29	65·45
5 7 5 8	379	152·59	69·21
5 8 5 9	468	157·76	71·56
5 9 5 10	368	166·40	75·48
5 10 5 11	348	170·86	77·50
5 11 6 0	245	177·45	80·49
6 0 6 1	326	218·66	99·18
Total ...	3000	147·86	67·07

He is here probably quoting Quetelet, as he did when he corrected for clothing by taking its weight to be 1/18 of the total weight in the case of males and 1/24 in the case of females. Hutchinson's conclusion was that an excess of 7 % or more over the normal weight for any given height would influence the vital capacity, but that a smaller excess would have no effect.

He treated age in the same way as weight for different height groups. He first examined 1088 subjects with no definite result. Then from a series of 1775 healthy subjects he found that from 15 to 35 years the vital capacity increased but after the age of 35 decreased by 19 cu. in., 11 cu. in., and 13 cu. in. respectively in the three succeeding 10-year periods, i.e. approximately at the rate of 117 c.c. per five years.

As regards chest, Hutchinson said:

"I do not find there exists any direct relation between the circumference of

the chest and the vital capacity; for the height of 5 feet 6½ inches the difference in vital capacity for an increase of ½ inch in the circumference is quite irregular."

He therefore rejected absolutely the circumference of the chest as a guide to vital capacity. This is in spite of the fact that he asserted that the most remarkable relation of the circumference of the chest was to weight, with which it increased in an *exact* arithmetical progression of 1 inch to every 10 lbs. How such a definite relation could exist between chest and weight, in conjunction with the relation he found between weight and vital capacity, without any relation between chest and vital capacity, Hutchinson did not attempt to explain.

To sum up his conclusions. In healthy cases vital capacity is chiefly affected by height, weight and age thus:

Height. An increase of 8 cubic inches at 60° for every inch of height.

Weight. At height 5 feet 6½ inches vital capacity is not affected under 161 lbs., but above this point it diminishes in the relation of 1 cubic inch per lb. up to 196 lbs. At other heights 10 % may be added to the mean height before applying this rule.

Age. A decrease of rather more than 1 cubic inch per year.

Chest. No relation.

Hutchinson also asserted that standing height was not correlated with sitting height, but that the latter was constant. He only gave figures for one pair of cases, but said that others gave the same result.

He considered that vital capacity was governed by the mobility of the chest rather than its absolute volume, but did not go into this point statistically beyond examining the lung capacities of 20 corpses.

He found that position might make a difference of 655 cubic centimetres in vital capacity. In support of his claim that vital capacity might be a useful measure to detect progress of disease, he gave the vital capacity in 22 early cases of phthisis and 9 advanced compared with the theoretical healthy values, calculated from his means of people of the same physical development (see Table IV).

Not many observations appear to have been published concerning the vital capacity of diseased persons, but Hutchinson's differences were larger than those found by other writers quoted below.

Hutchinson's work was of much statistical value. He did not, unfortunately, present his data in a form to allow correlation coefficients to be calculated, for one cannot get the standard deviation of his vital capacity, but, as we shall see later, in spite of his faulty method of getting means, the regression coefficients at which he arrived differ very little from those we have found in a sample of about 1000 Air Force cadets.

Shortly after the publication of Hutchinson's work, Fabius of Amsterdam, anticipating a criticism reiterated in our own time, objected that stem length and not height was the appropriate independent variable to which vital capacity

ought to be related. We have not seen Fabius' paper, which was apparently a degree thesis, but his statistics are re-printed in the monograph of Arnold.

TABLE IV.

Vital Capacities of Tuberculous Subjects.

MALES.

	Hutchinson.				Garvin, Lundsgaard and van Slyke.			
	Vital Capacity				Vital Capacity			
	Diseased (Observed) c.c.	Healthy (Calculated) c.c.	Difference c.c.		Diseased (Observed) c.c.	Class	Healthy (Calculated) from Dreyer c.c.	Difference c.c.
Early stage	1852	3605	1753	Nine incipient cases	3813	B	3734	+ 79
	1884	2835	951		3494	B	3991	497
	1721	2835	1114		3879	B	4012	133
	2130	3343	1213		3611	B	3991	380
	2098	3605	1507		3838	B	3970	132
	1966	3753	1787		3649	B	4054	405
	1639	3163	1524		4295	B	4137	+158
	2294	4031	1737		5104	B	4364	740
	1639	3343	1704		4095	B	4054	+ 41
	1803	3605	1802	Thirteen moderately advanced cases	2405	B	3537	1132
	2229	3753	1524		2986	B	4033	1047
	2212	3343	1131		3600	B	3884	284
	3146	3769	623		2894	B	4075	1181
	3687	4916	1229		3401	B	4262	861
	2376	3605	1229		3013	B	4096	1083
	3277	3933	656		3741	B	4075	334
	3032	3769	737		2749	B	4075	1326
	3572	3933	361		2952	B	4075	1123
	2114	3605	1491		4546	A	4825	279
	5637	7112	1475		3501	B	3842	341
	3605	4261	656		3894	A	4780	886
	3212	4162	950		3950	B	4179	229
Advanced stage	967	2212	1245	Nine advanced cases	2106	B	3514	1408
	1458	3671	2213		2606	B	3603	997
	1770	4162	2392		2696	B	3927	1231
	1180	2212	1032		2607	B	3777	1170
	1311	3753	2442		2100	A	4486	2386
	1229	4162	2933		4256	B	4033	+223
	557	4031	3474		2598	B	3906	1308
	2802	4424	1622		2449	A	4262	1813
	983	3884	2901		2400	B	4012	1612

From an analysis of these (Table V) it is evident that Fabius had no justification for his conclusion that stem length was a better variable to choose, and it also appears that stem length is not even approximately constant.

In 1858, Arnold of Heidelberg published a full monograph on the question, confirming generally Hutchinson's views, but attaching more importance than the latter did to chest circumference and expansion. From Arnold's tabulation of 216 adults (observations of his own, together with those of Fabius and Simon) we

find the correlation of stature and vital capacity to be $0{\cdot}7118 \pm 0{\cdot}0226$. The mean vital capacity was 3588 c.c. with a standard deviation of 487 c.c., giving a coefficient of variation of 13·6 %. The correlation is higher, but not significantly higher, than we have found. The variability of vital capacity is remarkably constant in all the series we have examined, despite differences both of methods of measurement and of mean values; Arnold's, Schuster's and our own results are nearly identical. Arnold also published data respecting women, with these we shall not deal in this paper.

TABLE V.

Fabius' Amsterdam Students.

	Mean	Standard Deviation	Coefficient of Variation	Number of Observations
Vital Capacity ...	3717·74 c.c.	505·90 c.c.	13·61	116
Standing Height ...	173·43 cms.	6·68 cms.	3·85	116
Sitting Height ...	75·22 cms.	4·38 cms.	5·82	116

Correlation Coefficients and Ratios.

	r	η	$\eta^2 - r^2$
Vital Capacity and Standing Height...	·661 ± ·035	·715 ± ·031	·0738 ± ·034
Vital Capacity and Sitting Height ...	·604 ± ·040	·652 ± ·036	·0600 ± ·031
Standing Height and Sitting Height...	·732 ± ·029	·740 ± ·028	·0124 ± ·014

Regression Equations	Equation	Partial Standard Deviation
Vital Capacity in c.c. from Standing Height in cms.	$x = 50{\cdot}0094z - 4955{\cdot}19$	379·71
Vital Capacity in c.c. from Sitting Height in cms.	$x = 69{\cdot}8229y - 1534{\cdot}28$	403·10

After Arnold's time several other papers were published, but we have given typical results and now pass to Dr Edgar Schuster's paper in *Biometrika*, VIII. 1911, p. 40), wherein the vital capacity and other dimensions of 959 Oxford undergraduates are analysed.

Schuster studied the correlations of vital capacity with stature, weight and strength of pull. (These are given in Table VI for comparison.) Schuster does not give any measure of the linearity of these correlations. The mean of these Oxford undergraduates is greater than that of any of Hutchinson's healthy classes, and especially greater than that of the class he calls "gentlemen." Schuster's vital capacities are the *mean* of each individual's attempts, most other investigators take the *maximum*. Schuster's results agree closely with those we obtained

from the Air Force data. The size of the standard deviations is to be noted in view of the suggested use of vital capacity as a test for the early diagnosis of disease. Though it might be argued that these data contain unfit persons as well as fit, it will be seen that standard deviations of the same size were obtained from a perfectly fit sample of Air Force cadets.

TABLE VI.

Schuster's Results. (Oxford Undergraduates, *Biometrika*, Vol. VIII.)

(Weight taken fully dressed except boots and coats.)
(Lung Capacity = Vital Capacity = Mean of 3 trials.)

Lung Capacity				Weight		
Age	No.	Mean c.c.	Standard Deviation c.c.		Mean st. lbs. oz.	Standard Deviation st. lbs. oz.
18	129	4184 ± 34	566·2 ± 23·8	General Average	10 11 15	1 2 6
19	330	4278 ± 22	602·5 ± 15·8			
20	209	4268 ± 29	611·6 ± 20·2	Stature		
21	137	4438 ± 39	683·1 ± 27·8			
22	95	4301 ± 37	540·9 ± 26·5	Mean		Standard Deviation
23 and over	59	4418 ± 59	674·8 ± 41·9			
General Average	959	4315	613·2	General Average	1765·0 mm.	66·08 mm.

Correlation Coefficients.

Age	Lung Capacity and Stature	Lung Capacity and Weight	Lung Capacity and Strength of Pull	Stature and Weight
18	·44 ± ·05	·55 ± ·04	·44 ± ·05	·50 ± ·04
19	·65 ± ·02	·62 ± ·02	·40 ± ·03	·63 ± ·02
20	·55 ± ·03	·52 ± ·03	·37 ± ·04	·68 ± ·03
21	·64 ± ·03	·66 ± ·03	·32 ± ·05	·76 ± ·02
22	·59 ± ·04	·62 ± ·04	·31 ± ·06	·72 ± ·03
General Average	·57	·59	·37	·66

Pearson's 1000 Cambridge Undergraduates, 1899 ·49 ± ·02

Garvin, Lundsgaard and van Slyke. These three authors published a paper entitled "Studies of Lung Volume,"* in 1918, which contains some detailed results on tuberculous subjects. The vital capacities for males deduced from their figures are given in Table IV. In the second column has been added the normal vital capacity from weight according to Dreyer's Tables, and the third column

* *Journal of Experimental Medicine*, XXVII. p. 87.

shews the differences. The loss of vital capacity shewn by all the incipient cases is well within the sampling error given by Schuster, and it is only in the more advanced cases, which have well developed physical signs of disease, that the difference can have any significance.

In 1919, Professor Georges Dreyer entered the field. He maintained that Hutchinson's work was fundamentally sound, but that (*a*) stem length should be substituted for standing height, (*b*) that the variables used should be connected by equations of the form $y = ax^b$, where a and b are constants. Dreyer was originally led to test this form of relationship by biological considerations. From a study of the relationship between body weight and aortic and tracheal area in animals, he was led to surmise that the constants were proportional to body surface. His position appears to be that, if a formula capable of a biological interpretation is found to describe a range of facts with tolerable efficiency—even if not actually so efficient as an *ad hoc* method—if, further, the range covered is wide, then such a method deserves the preference.

The practical results of Dreyer's investigations are summarised in a small volume of tables entitled "The Assessment of Physical Fitness" by G. Dreyer and G. F. Henson. The precise data and groupings used to compute the constants have not, so far as we know, been published, but are stated to depend upon "a number of observations sufficient to ensure a degree of accuracy that should prove entirely satisfactory." Tables for the computation of:

Normal Weight from stem length,
Normal Weight from circumference of the chest,
Normal Vital Capacity from weight,
Normal Vital Capacity from stem length,
Normal Vital Capacity from chest circumference,

are provided; there are separate tables for males and females, and both imperial and metric units are entered.

The instructions for using the tables state that one should first ascertain whether an individual's weight is normal, judged by his chest circumference and stem length. If it is so, the vital capacity-weight table should be used; if it is not, the values from the vital capacity-stem length and vital capacity-chest circumference are to be averaged. It may be remarked that, according to these tables, a variation of 10 °/₀ from the mean weight is probably, and of 15 °/₀, certainly abnormal. We have found that in a sample of 950 passed fit candidates for the Air Force, the coefficient of variation was 11·4 °/₀, while for constant chest circumference it was only reduced to 8·4 °/₀.

Dreyer's results have found support and criticism in various papers; we have no concern here either with supporters or critics, our problem being a purely biometric one, viz. whether in fact the formulae used for the tables or similar formulae were better descriptions of a sample population than those based upon elementary biometric considerations.

We knew that, owing to the energy of Wing Commander Martin Flack, C.B.E., a considerable mass of data suitable for our purpose had been collected in the Royal Air Force. By the kind permission of the Director of Medical Services of the Royal Air Force, and with the cordial co-operation of Wing Commander Flack—to whose expert advice and criticism we owe much—we have been able to use a large number of records, to which we now turn.

The data were measurements of 1238 candidates for commissions in the Royal Air Force, supplied by Wing Commander Flack. This population, distinguished as *A* in the tables, was analysed and the whole analysis repeated upon the 950 measurements relating to candidates passed as fit, tables relating to the latter being marked *B*. The selection has made very little difference in the arithmetical values of the constants. It will be seen that the mean and standard deviation of vital capacity are slightly larger than and the coefficient of variation almost identical with Dr Schuster's findings on Oxford undergraduates (see Tables VI and VII). There is a similarly close agreement in respect of stature, our men are slightly shorter but have the same variability. Dr Schuster's coefficient of variation was 3·74, ours 3·64. It may be said that, in respect of the variables studied, the Air Force candidates are in *pari materia* with the Oxford undergraduates, who provided some of the evidence upon which Dreyer relied.

From the table of coefficients of correlation (Table VIII) we may infer that few of the total regressions are strictly linear, but that the departures from linearity although significant are not large, excepting those instances into which weight and age enter. Passing to the effect upon variability of vital capacity produced by allowance for the other variables, it is found that the fourth order standard deviation is for *A* 496 c.c., 77·4 °/₀ of the standard deviation of zero order, for *B* data it is 76·6 °/₀ of the zero order value. This result is itself enough to shew that predictions from all four measurements, standing height, sitting height, weight and chest circumference, are subject to a very large error of sampling, roughly 10 or 12 °/₀ of the average value.

Returning to the deviation from linearity, we remark that all the deviations from the linear trend of importance occurred where observations were scanty. Apart from age (not at first considered owing to the limited range) standing height, chest circumference, and weight were least linear and, in the two latter cases, equations of type $y = ax^b$ were applied, but neither shewed a better graduation; vital capacity in terms of weight was, however, better graduated by a cubic than by a linear equation. The trend of means of vital capacity in terms of weight, actually shews the same apparent inflexion at higher weights mentioned by Hutchinson, but we can put little stress upon our result since the frequency of heavy weights amongst the cadets was small. It is, however, possible, as Hutchinson thought, that after a certain point increase of weight—due to mere adiposity—is unfavourable to vital capacity. The American actuaries found that "over weights" were worse lives on the average than "under weights."

When exponential functions were fitted to the data, we first obtained an

TABLE VII. *Air Force Candidates.*

	Mean	Standard Deviation	Coefficient of Variation	β_1	β_2
A (All Candidates) 1238 men:					
Vital Capacity	4569·72 c.c.	640·69 c.c.	14·02	·02835 ± ·011	3·1295 ± ·127
Weight	63·44 kg.	7·40 kg.	11·67	·63302 ± ·143	4·3434 ± ·520
Standing Height	174·82 cms.	6·36 cms.	3·64	·000299 ± ·000016	3·4933 ± ·206
Sitting Height	90·39 „	3·12 „	3·45	·000046 ± ·000016	3·0654 ± ·105
Chest	86·25 „	4·74 „	5·50	·1857 ± ·055	3·5411 ± ·256
Age	23·92 yrs.	5·07 yrs.	21·19	1·8657	5·8886
B (Fit Candidates) 950 men:					
Vital Capacity	4604·82 c.c.	614·72 c.c.	13·35	·0737 ± ·034	3·3746 ± ·230
Weight	63·48 kg.	7·24 kg.	11·41	·6098 ± ·179	4·4919 ± ·823
Standing Height	174·81 cms.	6·28 cms.	3·59	·000557 ± ·00030	3·3876 ± ·202
Sitting Height	90·34 „	3·06 „	3·39	·001447 ± ·000039	3·144 ± ·136
Chest	86·29 „	4·65 „	5·39	·1805 ± ·0058	3·4518 ± ·234
Age	23·80 yrs.	4·60 yrs.	19·42	1·7925	6·4425

TABLE VIII. *Air Force Candidates. Total Correlation Coefficients and Ratios.*

	A (All Candidates).			*B* (Fit Candidates).		
	r	η	$\eta^2 - r^2$	r	η	$\eta^2 - r^2$
Vital Capacity and Standing Height ...	·568 ± ·013	·588 ± ·013	·0229 ± ·0058	·586 ± ·014	·615 ± ·014	·0348 ± ·0068
Vital Capacity and Sitting Height ...	·527 ± ·014	·527 ± ·014	·00047 ± ·00084	·549 ± ·015	·554 ± ·015	·0051 ± ·0031
Vital Capacity and Weight	·483 ± ·015	·528 ± ·014	·0454 ± ·0082	·495 ± ·017	·531 ± ·016	·0377 ± ·0085
Vital Capacity and Chest	·413 ± ·016	·436 ± ·016	·012 ± ·0042	·395 ± ·019	·435 ± ·018	·0334 ± ·0080
Vital Capacity and Age	− ·0182 ± ·019	·211 ± ·019	·0440 ± ·0080	− ·0119 ± ·022	·190 ± ·021	·0360 ± ·0083
Weight and Chest	·750 ± ·009	·751 ± ·009	·00198 ± ·0017	·7405 ± ·0099	·7406 ± ·0099	·00011 ± ·00045
Sitting Height and Standing Height ...	·676 ± ·010	·700 ± ·010	·0336 ± ·007	·720 ± ·011	·736 ± ·010	·0224 ± ·0065
Weight and Sitting Height	·528 ± ·014	·528 ± ·014	·000 ± ·000	·540 ± ·016	·540 ± ·016	·000 ± ·000
Standing Height and Weight	·547 ± ·013	·582 ± ·013	·0390 ± ·008	·544 ± ·015	·580 ± ·015	·0401 ± ·0088
Chest and Sitting Height	·365 ± ·017	·367 ± ·017	·0013 ± ·0018	·367 ± ·019	·371 ± ·019	·0028 ± ·0023
Standing Height and Chest	·343 ± ·017	·357 ± ·017	·0099 ± ·0038	·336 ± ·019	·429 ± ·018	·0714 ± ·0117
Chest and Age	·302 ± ·017	·331 ± ·018	·0185 ± ·0052	·272 ± ·020	·312 ± ·020	·0232 ± ·0067
Weight and Age	·209 ± ·018	·241 ± ·019	·0145 ± ·0046	·178 ± ·021	·229 ± ·021	·0206 ± ·0031
Sitting Height and Age	·079 ± ·019	·150 ± ·019	·0162 ± ·0049	·053 ± ·022	·163 ± ·021	·0238 ± ·0068
Standing Height and Age	− ·0339 ± ·019	·179 ± ·019	·0308 ± ·0067	− ·0546 ± ·022	·218 ± ·021	·0446 ± ·0092

approximate value by fitting the logarithmic line by least squares and then improved the value by the use of one or other of the following methods, the use of which was suggested respectively by Dr Isserlis* and Mr Trachtenberg.

(I) To fit $y = \alpha x^\beta$. First find b an approximation to β by fitting the logarithmic equation by least squares. Then find the least square equations for making $\Sigma_{r=1}^{n} (y_r - \alpha x_r^\beta)^2$ a minimum, eliminate α from them and in the resulting equation for β, replace β by $b + u$, expand x^β in powers of $u \log x$ and solve for u neglecting u^2 and higher powers. Having got β, α can be obtained from the least square equations.

Thus notation

$$p_1 = \sum_{r=1}^{n} \log y_r, \qquad q_1 = \Sigma x^b, \qquad s_1 = \Sigma x^\beta,$$

$$q_2 = \Sigma x^{2b}, \qquad s_2 = \Sigma x^{2\beta},$$

$$p_2 = \sum_{r=1}^{n} \log x_r, \qquad q_3 = \Sigma y x^b, \qquad s_3 = \Sigma y x^\beta,$$

$$p_3 = \sum_{r=1}^{n} (\log x_r)^2, \qquad q_4 = \Sigma y x^b \log x, \qquad s_4 = \Sigma y x^\beta \log x.$$

$$q_5 = \Sigma x^{2b} \log x,$$

$$p_4 = \sum_{r=1}^{n} (\log x_r \log y_r). \qquad q_6 = \Sigma y x^b (\log x)^2,$$

$$q_7 = \Sigma x^{2b} (\log x)^2.$$

Make $\sum_{r=1}^{n} (\log y_r - \log a + b \Sigma \log x_r)^2$ a minimum.

This gives $$b = \frac{np_4 - p_1 p_2}{np_3 - p_2^2} \log \alpha = \frac{p_1 p_3 - p_2 p_4}{np_3 - p_2^2} = \frac{p_1 - bp_2}{n} = \frac{p_4 - bp_3}{p_2}.$$

Now fit $y = \alpha x^\beta$, i.e. make $\sum_{r=1}^{n} (y_r - \alpha x_r^\beta)^2$ a minimum.

$$\therefore \quad \Sigma x_r^\beta (y_r - \alpha x_r^\beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\Sigma x_r^\beta \log x_r (y_r - \alpha x_r^\beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or $$\alpha \Sigma x_r^{2\beta} = \Sigma y_r x_r^\beta \text{ and } \alpha \Sigma (x_r^{2\beta} \log x_r) = \Sigma (y_r x_r^\beta \log x_r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminate α. $$\therefore \quad \frac{\Sigma (x_r^{2\beta} \log x_r)}{\Sigma (x_r^{2\beta})} = \frac{\Sigma (y_r x_r^\beta \log x_r)}{\Sigma (y_r x_r^\beta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Take $\beta = b + u$ and solve (4) for u neglecting u^2 and expanding x^β.

$$\therefore \quad x^\beta = x^b (1 + u \log x),$$

$$x^{2\beta} = x^{2b} (1 + 2u \log x).$$

$$\therefore \quad \frac{\Sigma (x^{2b} \log x) + 2u \Sigma \{x^{2b} (\log x)^2\}}{\Sigma (x^{2b}) + 2u \Sigma \{x^{2b} \log x\}} = \frac{\Sigma \{y x^b \log x\} + u \Sigma \{y x^b (\log x)^2\}}{\Sigma (y x^b) + u \Sigma (y x^b \log x)},$$

or $$\frac{q_5 + 2uq_7}{q_2 + 2uq_5} = \frac{q_4 + uq_6}{q_3 + uq_4}, \quad \therefore u = \frac{q_3 q_5 - q_2 q_4}{q_2 q_6 + q_4 q_5 - 2q_3 q_7}.$$

From (3) $\alpha = \frac{s_3}{s_2} = \frac{s_4}{s_5}$, and if these two values of α agree, then the values of α and β are correct.

[* The process is Gauss' fundamental method applied to a special case. Ed.]

If more accurate values are wanted, repeat the process, using β instead of b.

(II) To fit $y = bc^x$ etc. or similar index forms.

Let y_x = observed value, $f(x, a, b, \ldots)$ = the theoretical value.

Let $y_x - f(x, a, b, \ldots) = \epsilon_x$.

Then
$$\log f(x, a, b, \ldots) = \log (y_x - \epsilon_x)$$
$$= \log y_x + \log \left(1 - \frac{\epsilon_x}{y_x}\right)$$
$$= \log y_x - \frac{\epsilon_x}{y_x} - \frac{1}{2}\left(\frac{\epsilon_x}{y_x}\right)^2 - \ldots \text{ if } \frac{\epsilon_x}{y_x} < 1.$$

TABLE IX A.

Air Force Candidates. Total and Partial Correlation Coefficients.

A (All Candidates).

	Vital Capacity and Standing Height	Vital Capacity and Sitting Height	Vital Capacity and Weight	Vital Capacity and Chest	Vital Capacity and Age
Total Coefficients	·568 ± ·013	·527 ± ·014	·483 ± ·015	·413 ± ·016	- ·0182 ± ·019
Partial Coefficients					
1st order. Variables kept constant:					
Standing Height	—	·235 ± ·018	·251 ± ·018	·282 ± ·018	—
Sitting Height	·338 ± ·017	—	·284 ± ·018	·279 ± ·018	—
Weight	·414 ± ·016	·365 ± ·017	—	·0879 ± ·019	—
Chest	·498 ± ·014	·443 ± ·015	·288 ± ·018	—	—
Age	·568 ± ·013	·530 ± ·014	·498 ± ·014	·439 ± ·015	—
2nd order:					
Standing Height and Sitting Height	—	—	·202 ± ·018	·248 ± ·018	—
Standing Height and Weight	—	·183 ± ·019	—	·152 ± ·019	—
Standing Height and Chest	—	·192 ± ·018	·073 ± ·019	—	—
Standing Height and Age	—	·238 ± ·018	·260 ± ·018	·299 ± ·017	—
Sitting Height and Weight	·276 ± ·018	—	—	·116 ± ·019	—
Sitting Height and Chest	·315 ± ·017	—	·128 ± ·019	—	—
Sitting Height and Age	·333 ± ·017	—	·304 ± ·017	·315 ± ·017	—
Weight and Chest	·430 ± ·016	·372 ± ·017	—	—	—
Weight and Age	·400 ± ·016	·364 ± ·017	—	·123 ± ·019	—
Chest and Age	·485 ± ·015	·444 ± ·015	·287 ± ·018	—	—
3rd order:					
Standing Height, Sitting Height and Weight	—	—	—	·153 ± ·019	—
Standing Height, Sitting Height and Chest	—	—	·041 ± ·019	—	—
Standing Height, Sitting Height and Age	—	—	·217 ± ·018	·273 ± ·018	—
Sitting Height, Weight and Chest	·292 ± ·018	—	—	—	—
Sitting Height, Weight and Age	·258 ± ·018	—	—	·134 ± ·019	—
Sitting Height, Chest and Age	·295 ± ·018	—	·127 ± ·019	—	—
Weight, Chest and Age	·404 ± ·016	·367 ± ·017	—	—	—
Standing Height, Weight and Chest	—	·183 ± ·019	—	—	—
Standing Height, Weight and Age	—	·189 ± ·018	—	·138 ± ·019	—
Standing Height, Chest and Age	—	·202 ± ·018	·079 ± ·019	—	—
4th order:					
Standing and Sitting Height, Weight and Age	—	—	—	·141 ± ·019	—
Standing and Sitting Height, Chest and Age	—	—	·0453 ± ·019	—	—
Standing Height, Weight, Chest and Age	—	·191 ± ·018	—	—	—
Sitting Height, Weight, Chest and Age	·261 ± ·018	—	—	—	—

TABLE IX B.

Air Force Candidates. Total and Partial Correlation Coefficients.

B (Fit only).

	Vital Capacity and Standing Height	Vital Capacity and Sitting Height	Vital Capacity and Weight	Vital Capacity and Chest	Vital Capacity and Age
Total Coefficients	·586 ± ·014	·549 ± ·015	·495 ± ·017	·395 ± ·019	−·0119 ± ·022
Partial Coefficients					
1*st order. Variables kept constant :*					
Standing Height	—	·227 ± ·021	·259 ± ·020	·259 ± ·020	—
Sitting Height	·328 ± ·020	—	·282 ± ·020	·248 ± ·021	—
Weight	·434 ± ·018	·386 ± ·019	—	·048 ± ·022	—
Chest	·524 ± ·016	·473 ± ·017	·328 ± ·020	—	—
2*nd order :*					
Standing and Sitting Height	—	—	·214 ± ·021	·225 ± ·021	—
Standing Height and Weight...	—	·172 ± ·021	—	·111 ± ·022	—
Standing Height and Chest	—	·187 ± ·021	·111 ± ·022	—	—
Sitting Height and Weight	—	—	—	·0763 ± ·022	—
Sitting Height and Chest	—	—	·158 ± ·021	—	—
Weight and Chest	·444 ± ·018	·390 ± ·019	—	—	—
3*rd order :*					
Standing Height, Sitting Height and Weight	—	—	—	·110 ± ·022	—
Standing Height, Sitting Height and Chest...	—	—	·082 ± ·022	—	—
Sitting Height, Weight and Chest	·284 ± ·020	—	—	—	—
Standing Height, Weight and Chest... ...	—	·171 ± ·021	—	—	—

Neglecting squares of $\frac{\epsilon_x}{y_x}$, $\log f(x, a, b, \ldots) = \log y_x - \frac{\epsilon_x}{y_x}$.

$$\therefore \quad \epsilon_x = y_x \{\log f(x, a, b, \ldots) - \log y_x\}.$$

We now deviate from the general case, which would minimise $\Sigma \frac{\epsilon_x^2}{\sigma_{y_x}^2}$, and make $\Sigma\epsilon_x^2$ a minimum,

i.e. $$\Sigma [y_x \{\log f(x, a, b, \ldots) - \log y_x\}]^2 \text{ a minimum}^*.$$

Differentiate with respect to a, b, c, etc. and get the equations for the constants.

The results of these are shewn in Table XI (p. 332), the two methods give practically the same result in the case we have tested. Method II has the advantage of being much shorter in working, but Method I can be applied repeatedly if a still better value is wanted.

We only tested those cases where the departure from the linearity was such as to suggest that a logarithmic function might be more suitable; in other cases, graphs gave no reason to expect any improvement; further, our main object was to institute a comparison with Dreyer's published method. For this particular purpose a simpler method, suggested to us by Professor Karl Pearson, leads to

[* This method therefore amounts to fitting logarithms by least squares and weighting with the *squares* of the ordinates. If $Y_x = \log y_x$, we have $1/\sigma_{Y_x}^2 = \bar{y}_x^2/\sigma_{y_x}^2$ where $\bar{y}_x$ is mean y_x for given x. Hence if we could consider σ_{y_x} constant for all values of x, this result would be, *without regard to the approximations by which it was deduced*, the weighted logarithmic least square equation and might be written down at once. The assumption σ_{y_x} = constant would need justification. Ed.]

TABLE X.

Multiple Correlation Coefficients.

	A. All Candidates	*B.* Fit Candidates
One Variable.		
Vital Capacity from :		
Standing Height	·5679 ± ·013	·5858 ± ·014
Sitting Height	·5266 ± ·014	·5439 ± ·015
Weight	·4831 ± ·015	·4947 ± ·017
Chest	·4131 ± ·016	·3946 ± ·019
Age	−·01818 ± ·019	−·0119 ± ·022
Two Variables.		
Vital Capacity from :		
Standing Height and Chest	·6136*	·6246
Standing Height and Weight	·6042	·6222
Standing Height and Sitting Height	·6000	·6139
Sitting Height and Weight	·5792	·5977
Sitting Height and Chest	·5775	·5871
Standing Height and Age	·5679	—
Sitting Height and Age	·5300	—
Weight and Age	·4982	—
Weight and Chest	·4892	·4965
Chest and Age	·4395	·4098
Three Variables.		
Vital Capacity from :		
Standing Height, Sitting Height and Chest	·6321	·6392
Standing Height, Sitting Height and Weight	·6215	·6367
Standing Height, Chest and Age	·6189	—
Standing Height, Chest and Weight	·6163	·6283
Standing Height, Weight and Age	·6069	—
Standing Height, Sitting Height and Age	·6006	—
Sitting Height, Chest and Age	·5933	—
Sitting Height, Weight and Age	·5893	—
Sitting Height, Weight and Chest	·5869	·6008
Chest, Weight and Age	·5095	—
Four Variables.		
Vital Capacity from :		
Standing Height, Sitting Height, Chest and Age	·6390	—
Standing Height, Sitting Height, Chest and Weight	·6329	·6423
Standing Height, Weight, Chest and Age	·6325	—
Standing Height, Sitting Height, Weight and Age	·6252	—
Sitting Height, Weight, Chest and Age	·6017	—
Five Variables.		
Vital Capacity from :		
Standing Height, Sitting Height, Weight, Chest and Age	·6400	—

* The significance of these values of ρ of the second and higher orders may be judged by comparison with the following root-mean-square values of ρ from independent variables, calculated from Yule's formula $\rho_0 = \frac{(n-1)^{\frac{1}{2}}}{N^{\frac{1}{2}}}$, where n = number of variables and N = number of observations.

	A	*B*
Vital Capacity from 2 other variables	·0402	·0459
,, ,, 3 ,, ,,	·0492	·0562
,, ,, 4 ,, ,,	·0568	·0649
,, ,, 5 ,, ,,	·0636	·0725

instructive results. If the exponential relation be a more adequate expression of the facts than a linear relation, it follows that the correlation between y and x^n should be significantly greater than the correlation between x and y, where y is vital capacity and x a physical dimension. We have computed the correlation

TABLE XI.

Equations for finding Vital Capacity from Chest (see Table XIII A).
Different Methods of Fitting $x_2 = \alpha x_1^{\beta}$.

x_2 = Vital Capacity in cubic centimetres. x_1 = Chest in centimetres.

	χ^2	P
(1) Equation obtained by fitting a straight line to the logs $x_2 = 39{\cdot}41684 x_1^{1{\cdot}064471}$	39·021	·000977
(2) Equation obtained by Method I applied to (1) $x_2 = 44{\cdot}488878 x_1^{1{\cdot}039177}$ ...	33·507	·005045
(3) Equation obtained by Method II $x_2 = 44{\cdot}14914 x_1^{1{\cdot}041038}$	33·548	·005015

and correlation ratio on a random sample of 100 fit candidates with the results shewn in Table XII. It will be seen that the constants are sensibly equal. This result might have been deduced from the size of the coefficient of variation, since the correlation between y and x^n only differs from that between y and x by terms of the order $\left(\frac{\sigma_y}{\bar{y}}\right)^2$.

TABLE XII.

100 *Fit Air Force Candidates Chosen at Random.*

	r	η	$\eta^2 - r^2$
Vital Capacity and Weight ...	·4733 ± ·052	·5132 ± ·050	·0394 ± ·027
Vital Capacity and (Weight)$^{0{\cdot}72}$ ...	·4830 ± ·052	·5193 ± ·049	·0364 ± ·026

It has also been suggested that the high variability of the sample population is due to heterogeneity, i.e. to a mixture of two populations having different means and standard deviations for the characters considered. A mixed population so constituted would exhibit a greater relative variability than either constituents, it would also in this case, we think, exhibit a higher arithmetical correlation. Hence the correlations within the homogeneous sub-groups should be lower than the values we have obtained and, since the latter are inconsistent with the existence of a relation sufficiently stringent to give much weight to individual predictions, we are not, on the basis of the evidence before us, disposed to attach much importance to the point, although it should receive further consideration. It must also be noticed that the coefficient of variability of vital capacity is sensibly the same in the samples derived from such different populations as those of Schuster, Arnold and ourselves.

Returning to the main problem, we have applied the following experimental test.

All the regression equations of all four orders in both *A* and *B* (age not included) have been tested by substituting a random sample of 50 Air Force candidates *not* included in those from which the constants were calculated. The vital capacities of the same 50 were also calculated from each of Dreyer's Tables. The root-mean-square error of these results is shewn in Tables XIII A and B together with the theoretical standard deviations for each regression equation. In every case, the regression equations give a slightly smaller error than Dreyer's formulae. The observed errors of the regression lines are very near the theoretical ones, standing height and sitting height on the whole having more weight than weight or chest, but in one or two cases, owing to errors of sampling, a higher order equation gives a slightly larger observed error than a lower order one. The observed errors from the *B* equations, as would be expected, are on the whole smaller than those from the corresponding *A* equations. In both sets if only one variable is used, standing height gives the smallest theoretical error, if two are used, standing height and chest, if three are used, standing height, sitting height and chest. The theoretical errors in *B* range from 565 c.c. to 471 c.c. according to the different variables used to determine the vital capacity, and the corresponding observed mean errors range from 610 c.c. to 490 c.c. while those from Dreyer's equation range from 801 c.c. to 586 c.c.

Incidentally if we translate our total regression coefficients into English units to compare with Hutchinson, we find that, in spite of his errors, his final conclusions (except for sitting height) do not differ greatly from those obtained here, e.g.

Height. Hutchinson says an increase of 1 inch in standing gives an increase of 8 cubic inches in vital capacity.

By *A* an increase of 1 inch in standing height gives an increase of 8·88 cubic inches in vital capacity.

By *B* an increase of 1 inch in standing height gives an increase of cubic 8·87 inches in vital capacity.

Weight. Hutchinson says an increase of 1 lb. in weight gives an increase of 1 cubic inch in vital capacity.

By *A* an increase of 1 lb. in weight gives an increase of 1·16 cubic inches in vital capacity.

By *B* an increase of 1 lb. in weight gives an increase of 1·16 cubic inches in vital capacity.

Chest. Hutchinson says there is no direct relation between chest and vital capacity, but if we combine his rates for weight and chest, and weight and vital capacity it follows that an increase of 1 inch in chest gives increase of 10 cubic inches in vital capacity.

TABLE XIII A.

Air Force Data. Regression Equations. A (All Candidates).

x_1 = Chest in centimetres.
x_2 = Vital Capacity in cubic centimetres.
x_3 = Weight in kilogrammes.
x_4 = Standing Height in centimetres.
x_5 = Sitting Height in centimetres.

Total Standard Deviation of Vital Capacity = σ_2 = 640·69 c.c.

Vital Capacity from :		Vital Capacity Theoretical Standard Deviation c.c.	Square Root of Mean Square Error observed from a random 50	
			Regression Equation c.c.	Dreyer's Tables c.c.
1st order :				
Standing Height	$x_2 = 57{\cdot}239x_4 - 5436{\cdot}92$ ... ... Partial	$\sigma_{2{\cdot}4}$ 527·36	527	—
Sitting Height ...	$x_2 = 108{\cdot}109x_5 - 5202{\cdot}51$	$\sigma_{2{\cdot}5}$ 544·67	527	586
Weight	$x_2 = 41{\cdot}807x_3 + 1917{\cdot}30$	$\sigma_{2{\cdot}3}$ 560·98	561	726
Chest	$x_2 = 55{\cdot}784x_1 - 241{\cdot}77$	$\sigma_{2{\cdot}1}$ 583·46	605	801
2nd order :				
Standing Height and Chest ...	$x_2 = 33{\cdot}396x_1 + 48{\cdot}682x_4 - 6821{\cdot}39$	$\sigma_{2{\cdot}14}$ 505·92	523	—
Standing Height and Weight ...	$x_2 = 21{\cdot}321x_3 + 43{\cdot}674x_4 - 4418{\cdot}10$	$\sigma_{2{\cdot}34}$ 510·52	513	—
Standing Height and Sitting Height ...	$x_2 = 39{\cdot}336x_4 + 53{\cdot}968x_5 - 7185{\cdot}46$	$\sigma_{2{\cdot}45}$ 512·53	503	—
Sitting Height and Weight	$x_2 = 24{\cdot}592x_3 + 77{\cdot}280x_5 - 3975{\cdot}99$	$\sigma_{2{\cdot}35}$ 522·28	510	646
Sitting Height and Chest	$x_2 = 34{\cdot}407x_1 + 89{\cdot}005x_5 - 6443{\cdot}38$	$\sigma_{2{\cdot}15}$ 523·04	514	670
Weight and Chest	$x_2 = 15{\cdot}701x_1 + 34{\cdot}262x_3 + 1041{\cdot}76$	$\sigma_{2{\cdot}13}$ 558·81	566	760
3rd order :				
Standing Height, Sitting Height and Chest	$x_2 = 29{\cdot}113x_1 + 35{\cdot}476x_4 + 43{\cdot}117x_5 - 8040{\cdot}82$ ...	$\sigma_{2{\cdot}145}$ 496·47	514	—
Standing Height, Sitting Height and Weight	$x_2 = 17{\cdot}307x_3 + 32{\cdot}318x_4 + 41{\cdot}933x_5 - 5968{\cdot}49$ ...	$\sigma_{2{\cdot}345}$ 501·94	493	—
Standing Height, Weight and Chest ...	$x_2 = 24{\cdot}927x_1 + 8{\cdot}516x_3 + 45{\cdot}434x_4 - 6063{\cdot}37$ ...	$\sigma_{2{\cdot}134}$ 504·57	517	—
Sitting Height, Weight and Chest	$x_2 = 19{\cdot}368x_1 + 15{\cdot}002x_3 + 78{\cdot}549x_5 - 5152{\cdot}81$ ...	$\sigma_{2{\cdot}135}$ 518·74	510	683
4th order :				
Standing Height, Sitting Height, Weight and Chest	$x_2 = 24{\cdot}577x_1 + 4{\cdot}721x_3 + 34{\cdot}164x_4 + 41{\cdot}525x_5 - 7575{\cdot}83$ ...	$\sigma_{2{\cdot}1345}$ 496·06	507	—

TABLE XIII B.

Air Force Data. Regression Equations. B (Fit Candidates).

x_1 = Chest in centimetres.
x_2 = Vital Capacity in cubic centimetres.
x_3 = Weight in kilogrammes.
x_4 = Standing Height in centimetres.
x_5 = Sitting Height in centimetres.

Total Standard Deviation of Vital Capacity = σ_2 = 614·72 c.c.

Vital Capacity from :		Vital Capacity Theoretical Standard Deviation c.c.	Square Root of Mean Square Error observed from a random 50	
			Regression Equation c.c.	Dreyer's Tables c.c.
1st order :				
Standing Height	$x_2 = 57{\cdot}314x_4 - 5588{\cdot}55$ ... ... Partial	$\sigma_{2{\cdot}4}$ 498·20	556	—
Sitting Height ...	$x_2 = 110{\cdot}251x_5 - 5355{\cdot}29$	$\sigma_{2{\cdot}5}$ 513·67	528	586
Weight ...	$x_2 = 41{\cdot}997x_3 + 1938{\cdot}69$	$\sigma_{2{\cdot}3}$ 534·23	530	726
Chest ...	$x_2 = 52{\cdot}122x_1 + 107{\cdot}44$	$\sigma_{2{\cdot}1}$ 564·85	610	801
2nd order :				
Standing Height and Chest ...	$x_2 = 29{\cdot}429x_1 + 49{\cdot}986x_4 - 6672{\cdot}76$	$\sigma_{2{\cdot}14}$ 481·21	517	—
Sitting Height and Weight ...	$x_2 = 21{\cdot}220x_3 + 44{\cdot}008x_4 - 4435{\cdot}43$	$\sigma_{2{\cdot}34}$ 481·23	512	—
Standing Height and Sitting Height	$x_2 = 38{\cdot}664x_4 + 53{\cdot}122x_5 - 6953{\cdot}214$	$\sigma_{2{\cdot}45}$ 485·24	500	—
Sitting Height and Weight ...	$x_2 = 23{\cdot}753x_3 + 79{\cdot}952x_5 - 4125{\cdot}94$	$\sigma_{2{\cdot}35}$ 492·83	510	646
Sitting Height and Chest ...	$x_2 = 29{\cdot}434x_1 + 93{\cdot}821x_5 - 6410{\cdot}67$	$\sigma_{2{\cdot}15}$ 497·63	510	670
Weight and Chest ...	$x_2 = 8{\cdot}251x_1 + 38{\cdot}075x_5 - 1475{\cdot}70$	$\sigma_{2{\cdot}13}$ 533·60	559	760
3rd order :				
Standing Height, Sitting Height and Chest	$x_2 = 25{\cdot}426x_1 + 35{\cdot}862x_4 + 43{\cdot}068x_5 - 7749{\cdot}10$	$\sigma_{2{\cdot}145}$ 472·76	491	—
Standing Height, Sitting Height and Weight	$x_2 = 17{\cdot}641x_3 + 32{\cdot}093x_4 + 40{\cdot}327x_5 - 5768{\cdot}61$	$\sigma_{2{\cdot}345}$ 474·04	494	—
Standing Height, Weight and Chest ...	$x_2 = 17{\cdot}277x_1 + 12{\cdot}424x_3 + 45{\cdot}221x_4 - 5579{\cdot}95$	$\sigma_{2{\cdot}134}$ 478·33	500	—
Sitting Height, Weight and Chest	$x_2 = 12{\cdot}037x_1 + 17{\cdot}834x_3 + 80{\cdot}782x_5 - 4863{\cdot}83$	$\sigma_{2{\cdot}135}$ 491·41	506	683
4th order :				
Standing Height, Sitting Height, Weight and Chest ...	$x_2 = 16{\cdot}790x_1 + 9{\cdot}136x_3 + 33{\cdot}4136x_4 + 39{\cdot}860x_5 - 6866{\cdot}09$	$\sigma_{2{\cdot}1345}$ 471·12	490	—

By *A* an increase of 1 inch in chest gives an increase of 8·65 cubic inches in vital capacity.

By *B* an increase of 1 inch in chest gives an increase of 8·08 cubic inches in vital capacity.

Chest on Weight. Hutchinson says an increase of 10 lbs. in weight gives *exact* increase of 1 inch in chest.

By *A* an increase of 10 lbs. in weight gives an increase of ·858 inch (and is very nearly linear).

By *B* an increase of 10 lbs. in weight gives an increase of ·850 inch (and is very nearly linear).

From the results above described we draw the following conclusions.

(1) The relation between vital capacity taken as dependent and weight, height—however measured—and chest circumference taken as independent variables is rather better described by multiple linear regression formulae than by the "rational formulae" of Professor Dreyer, when the data under study are measurements of ostensibly normal adult males.

(2) The systematic use of the method of multiple regression has the advantage over Professor Dreyer's method of bringing out the extent to which the accuracy of an estimate of vital capacity is improved, by the introduction of more and more independent variables.

(3) It appears from such analysis that the variability of the vital capacity of normal male adults is very great and remains very great, being reduced by less than 25 $^{0}/_{0}$ of its initial value, when as many as four independent variables are used for estimation.

(4) Upon the evidence now available, it does not appear that either biometric or "rational" formulae can deduce from the non-physiological or ordinary anthropometric constants, estimates of normal vital capacity confined within sufficiently narrow limits to possess real prognostic value in individual cases.

(5) The substitution of stem length for height does not improve the accuracy of prediction and, with the minor exceptions noticed above, Hutchinson's method is as good as or better than its recent modifications.

In conclusion, we desire to express our sincere thanks to various friends, including Wing Commander Flack, Professor Karl Pearson and Professor Dreyer for advice and criticism.

It is likewise our painful duty to record that a valued friend and colleague, whose strenuous work powerfully contributed to the execution of what has been a laborious task, has not been spared to collaborate in the published paper. The death of Miss E. C. C. Allen has not only deprived us of a valued friend and colleague, it has deprived biometric science of a brilliant worker.

A STUDY IN HUMAN FERTILITY.

By G. D. MAYNARD, M.D.

THE solution of many social and eugenic problems is dependent on an accurate knowledge of the laws of human fertility, but unfortunately data from which such laws may be directly determined have been scanty, and indirect methods of approaching the problem were of necessity, therefore, often adopted. The New Zealand Census Fertility Report of 1918 added a new and important set of figures, and it is an examination of some of these which is undertaken in this paper.

The following discussion is concerned chiefly with an attempt to discover the optimum age for marriage. The actual fertility-rate for each age at marriage, that is number of living children born to each married woman, is directly available from the tables. From the social point of view it is, no doubt, of equal importance to determine the effective fertility-rate, that is, the average number of children who survive to adult life. Unless, however, children born of women at certain ages are less likely to survive to adult life than those born of either younger or older mothers, the actual fertility-rate will probably be closely correlated with the effective rate, where the grouping is by age of mother at marriage, and class distinctions are not made. The age of maximum fecundity*, clearly, need not be identical with the best age for marriage in order that either the actual or the effective fertility-rate should be a maximum.

Marriage *de facto* is the social relationship which in this connection is of concern to the biologist, but for obvious reasons it is only the *de jure* marriages of which we get direct records. To overcome this difficulty Mr A. O. Powys (*Biometrika*, Vol. I) excluded from his figures women who bore a child within 9 months from the date of the recorded marriage, and called such unions "prejudiced" marriages. The problem he discussed was the "age of maximum fertility in Anglo-Saxon women in Australia," and he found it to be at about 25 years of age. He writes, "They show quite clearly that at any rate for the Anglo-Saxon race the view of Körösi that fertility is greater the younger the wife is not correct. The woman reaches her greatest reproductive vigour between 24·5 and 27 years, according to climate, and the man at about 32 years." I am unable to refer to Körösi's writings, but so far as the above extract is concerned, it seems possible that here two different meanings are attached to the word fertility, and this may explain the difference in views. Mr Powys very definitely states what he means in this sentence by the use of the phrase "her greatest reproductive vigour," but Körösi's view is

* Fecundity is used with the meaning of potential fertility, and fertility the degree of success in child-bearing. The "age of maximum fecundity" would connote the period in the life of a woman when exposure to risk would be most likely to be followed by pregnancy.

not quoted in sufficient detail to make it clear that he was referring to the same matter. It might be true, that all women reach an age of greatest reproductive vigour, say at 25 years, but this would not necessarily give us any clue to fertility in the sense of largest resultant family. In this latter sense of the term, it seems possible that the words accredited to Körösi in the above extract are approximately correct even for groups of Anglo-Saxon women, as the figures studied in this paper suggest.

The New Zealand Census Fertility Report (1918) gives valuable information in regard to this problem, but is necessarily defective in certain respects. While it is possible to reconstruct the tables, so that the number of children born to women at various ages at marriage can be determined, and the number surviving at the date of the Census (1916), clearly no information is available in regard to such proportion of each group as did not survive to the census year. Although, as will be shewn later, European women in New Zealand aged 15 at marriage have the greatest fertility; information as to what percentage of the original group the survivors represent is not available. It is possible that the survivors are a highly selected sample, selected in respect to the character here measured, and such selection may not be uniform with that of women married at other ages. This and other sources of fallacy, referred to later, must be borne in mind when deductions are drawn from the figures recorded.

The distributions given in Table I were compiled by extracting, for example, from the printed Tables arrays of families for women at ages 45, 46, etc. to 65 at date of census, and who had been married 30, 31, etc. years, and in this case giving when summed the distribution according to size of family for women all aged 15 at marriage, and who had lived to complete the average child bearing period, namely, to 45.

On plotting the data thus obtained, it was seen that no simple curve would give a good representation of the distributions of families for given age at marriage, unless some modification was made in the group "no children," and this becomes more and more evident as the distributions consist of women married at higher ages. It is clear that whether or no a *relative* sterility in the male be reflected in the number of children a woman bears, a complete male sterility must affect the zero group, and, therefore, this group will be a measure of both male and female sterility. While, therefore, it is certain that if the husband be sterile the woman to whom he is mated must fall into the "zero" group, it is not equally evident, having regard to the physiology of the human male reproductive process, and the frequent repetition of risk of conception, that relative sterility in the male will be a factor of much importance in modifying the fertility distribution of wives when completed families are considered. If this assumption be correct, then excluding the zero group, the distribution according to size of family is in the main a measure of female fertility, while adventitious factors are undoubtedly introduced into the group "no children." To include this group in the distribution without modification would therefore lead to erroneous values for the constants, if the object aimed at be

to obtain a measure of the fertility of women. Assuming as would seem probable, that the distribution according to size of family (including wives with no children) would be represented by one of the unimodal curves of the Pearson types, it is clear that the number of sterile women as recorded requires adjustment by removing those cases of sterility which are not due to the woman. No direct measure of the reduction could of course be obtained, but by inspection of a graph of the whole distribution an idea could be formed of what might be a reasonable number of zero families so that a simple curve would be a close representation of the data. The zero group was therefore adjusted by inspection and the constants of the distribution obtained from the recorded values of the other groups, plus the assumed value for the zero group. The equation to this curve having been found, and the values of the ordinates corresponding to 1, 2, 3, etc. children in a family calculated, the "goodness of fit" test was applied to all groups excepting the zero group. If the value of P, so obtained, did not give a reasonably satisfactory value, further adjustments to the data for the sterile wives was tried, previous results being used as a guide to subsequent alterations. The equation to the best fitting curve as found by this method was then adopted, and the value of the calculated ordinate for the zero group assumed to be the true measure of female sterility for the distribution of women under consideration. The constants of the distributions so treated are given in Table II, and in addition the mean size of the family calculated from the unadjusted data.

There are other factors which may affect the size of the 'zero' group, and which would not strictly be due to constitutional variation in degrees of female fertility on which in the main the type of the distribution probably depends; for example, certain surgical operations, acquired disease, and of course voluntary prevention of conception, all of which are in relation to a potential female fertility adventitious conditions.

There is little doubt that, in at least some of the distributions, slightly better fitting curves might be obtained by further modification in this group, but I do not think that such alterations are likely to affect the general deductions which can legitimately be drawn from the table as presented.

In the higher age groups, namely marriage at 37 and 40, some modification in the tail of the distribution was also necessary before a reasonable fit could be obtained. The unaltered distributions gave J-shaped curves with impossible starting points. Further justification for such alterations was provided by inspection of the returns for late marriages, as it was evident that misstatements occurred. For example, one woman is recorded as having been married at 47 and having had a family of 10 living children, a most unlikely event. It is probable that either the marriage recorded was a second marriage, or that the date of marriage recorded was merely the legal recognition of a pre-existing state. The earlier the age of marriage the less likely are misstatements of this kind to occur.

An inspection of Table II shews: first, that, within the range studied, whichever series of mean values be adopted, the younger the age at marriage the larger

the family; and secondly, that the type of the fertility curves changes from the symmetrical Type II for ages 15 and 17 to skew curves of Types I, III or VI at the higher ages. The correlation of age at marriage with size of family as obtained from Table I is

$$r = -\cdot 5068$$

and the correlation ratio

$$\eta = \cdot 5114.$$

Thus the regression is not strictly linear, although as seen in the left-hand bottom diagram of Fig. 2, p. 353, the divergency is not great.

That early marriage is detrimental to the woman and results in a restricted family and unhealthy children is a view widely held, although, so far as I can ascertain, one based rather on what are called "general principles" than on ascertained facts. On general principles, however, the reverse might equally be expected, for if it were really detrimental to a race that early conceptions should occur, the age of puberty should have become delayed through the process of evolution. Among the animals, and in some human societies, desire and fulfilment wait only on opportunity, so that it is not unreasonable to expect that the appearance of the sexual passions should coincide with the optimum age for marriage. An experienced stock breeder informs me that if the mating of Friesland heifers be delayed the calves are inferior and the mothers have smaller families, while among Shorthorn cattle early mating is detrimental to both the mother and offspring. If this be so, then it is clear that "general principles" are of little value as a guide in such problems, and that even within the limits of a genus a uniform law may not hold.

The analysis of the figures in the above tables seems to indicate that from the age of 15 onward the postponement of marriage results in a decrease in size of resultant family. In some figures very kindly sent me by Prof. Pearson dealing with similar data from England and Scotland, it appears that the largest families are associated in England with marriage at 16, and in Scotland at 18. The mean family of girls married at 16 in the New Zealand Returns is 8·178, slightly lower than that found for age 15.

Certain possible sources of fallacy, however, must be considered. First, if the girls who marry at the earlier ages are drawn from a super-fertile section of the population a disturbing factor of importance would have been introduced. It is generally believed that the lower social ranks are more fertile, and also marry at an earlier age, than the better educated classes. If the values here found can be accepted at their face value then increase in fertility is closely associated with age at marriage, and the higher fertility as a class characteristic may be more apparent than real. The comparatively undifferentiated character of the New Zealand population would, I think, for general population statistics, reduce such a disturbing factor to a minimum. Secondly, if strong development of sexual desire be correlated with fertility the early marriages may consist of a highly selected group. It is possible that this may be a factor of some importance, although one I do not see how to estimate or to check. Thirdly, if there be a selective mortality among the less fertile of the child-bearing women, those surviving to 45 or over will be

progressively less highly selected the later the age of marriage. If this be so a study of the number of children born within a limited period, after marriage at various ages, might indicate the importance of such a factor.

In Table VI is recorded the distribution of families of women who have completed 15 years of married life, and no more. Three groups are given, age at marriage 15—20; 21—25; and 26—30. Owing to the comparatively small numbers available for treatment in this way, the ages have been grouped so that errors of sampling may be reduced. Table VII gives the constants of these distributions, and it will be seen that the same order is maintained. It would seem, therefore, unlikely that a selective death-rate of the less fertile women is a factor of sufficient importance in any of the groups to disturb the general results obtained.

An inspection of the tables from which the family arrays for age at marriage were derived, brings to light another point of interest, namely, that the older women have on the average the larger families. Table III is given as an example, although the groups at all ages at marriage exhibit the same feature.

This table gives the distribution of families of women at ages 45 to 65 at date of census, all of whom were married at 20. Reference to the last column giving the mean family for each array, shews the general trend of the family to increase as the age of the mother increases, and this cannot to any material extent be due to additional births after the age of 45. The correlation of age at census with size of family for this group is

$$r = \cdot 2804 \pm \cdot 0092,$$

and the regression is closely linear,

$$y = \cdot 1743x.$$

Similar values for age 15 at marriage are

$$r = \cdot 1306 \pm \cdot 0385 \quad \text{and} \quad y = \cdot 0973x.$$

For age 25 at marriage,

$$r = \cdot 2506 \pm \cdot 0107 \quad \text{and} \quad y = \cdot 1306x.$$

There are at least four possible factors which must be considered in relation to this phenomenon:

(*a*) That the value found for "r" is the measure of the correlation of longevity and fertility.

(*b*) That there has been a progressive increase in prevalence of artificial restriction of families.

(*c*) That a biological decrease in fertility has been in operation, other than an intentional restriction.

(*d*) That improved hygienic conditions leading to a reduced infantile death-rate have lowered the birth-rate.

Professor Pearson* and his co-workers have shewn that, in certain groups of mothers studied by them, fertility was correlated with longevity. The correlation found for mothers in Series I was $r = \cdot 5009$ and in Series II $r = \cdot 2374$. Further, that

* *Proceedings of Royal Society*, Vol. LXVII. p. 159 et seq.

after the age of 50 the regression was approximately linear. These authors write: "fertility is correlated with longevity after the fecund period is passed."

The correlation values as recorded above fall into line with that found for Series II quoted above.

The statistical determination of the extent of an effective artificial restriction in the size of family will be difficult to obtain until the extent of other factors as agents in effecting maternal-fertility have been assessed. The linearity of the regression line of age at census on size of family, not only for age 20 at marriage, but for other ages, renders it, I think, improbable that artificial restriction is a factor of much importance in producing this result.

It is possible that as a result of change in social or other conditions a racial decrease in fertility may be taking place, apart from a conscious restriction of the family in certain classes of the people. If the group, aged 20 at marriage, be divided at age 54 the constants of the distribution of families for the older half of the table resemble those obtained from the whole table for the age of 17 at marriage. There is thus for this subgroup of women an apparent gain of nearly three years, which does, I think, suggest the possibility of a biological change in fertility having taken place. Change of type of the distribution, owing to elimination of the less fertile at the higher ages, would presumably be equally operative throughout both groups, 17 and 20. No doubt a deliberate restriction of the family might produce the same result, provided it be reasonable to assume that effective methods were sufficiently widely known and their use progressively desired by the general population of New Zealand wives, married in 1871 to 1891.

Improved hygienic conditions and care of the baby, if successful in decreasing the infantile mortality, might conceivably be an important factor in reducing the number of children born to a woman. A high infantile mortality in the first months of life reduces appreciably the duration of lactation, and therefore decreases the period of immunity to pregnancy which is usual during lactation. If, however, this were of importance in raising the mean family as determined by the number of children born, its extent would be indicated by comparing the mean number of children surviving infancy for each age of mother at marriage. Tables IV and V give the data in respect of children who survived to date of census, and therefore, in regard to the older mothers, will on the whole represent adults, and thus provide too stringent an elimination for this purpose. Nevertheless, even in these tables the older women are seen to have the larger families. Table IX corresponds to Table III, but records children living at date of census, $r = \cdot 1696 \pm \cdot 0097$ and $y = \cdot 0933x$. It does not seem probable therefore that a hygienic factor is one of much importance from this point of view.

Powys, in the paper already referred to, attempted to estimate the age of greatest fertility for both males and females in the Australian population. The method adopted is not perhaps ideal, but the best available with the data at his disposal. He assumed that the ratio of women who bore children in the 9th to the 12th month after marriage for groups of women married at different ages would be

an index of fertility at these ages. He had therefore to eliminate from his tables all cases of women who bore children before the 9th month, on the very reasonable assumption that in the great majority of such cases conception had antedated marriage. While it is clear that to have included "prejudiced" marriages would have been unsound practice, it is not equally clear that their exclusion did not leave selected groups of the relatively infertile at the earlier ages. Powys writes: "It must be borne in mind that most of the marriages contracted under the age of 20 are compulsory, i.e. were contracted after conception had followed illicit intercourse." If this be true of the Australian figures, it is not unlikely that a considerable proportion of those marriages considered as unprejudiced were nevertheless also compulsory owing to discovery, fear of pregnancy, and so forth, but as pregnancy did not follow they constitute to some extent a selection of the less fertile of the group. Pregnancy in the early months, especially in the unmarried girl, is extremely difficult for the untrained to recognise, and many mistakes would be sure to occur. An unmarried girl who has exposed herself to risk will probably regard herself as pregnant if she has missed a menstrual period, whereas false alarms of this nature are not uncommon. There are other objections that may be raised to these returns which are equally, or nearly equally, applicable to those discussed in this paper. The homogeneity of the population in regard to class fertility must be assumed, but should there be a variable class fertility and also a tendency for different classes to marry at different ages, an additional unmeasured complication will be introduced.

Further it may be doubted whether the 3 months period adopted is entirely satisfactory as a measure of age of greatest fecundity. There are many psychological and other factors which appear to influence the chance of conception taking place in the first few months of marriage, and such factors may not be uniform at all ages so that we may in reality be measuring the relationship of these factors to age, and therefore, a longer period would be preferable in order to obtain the true "age of greatest fertility."

The figures in the New Zealand Report do not enable a table similar to that of Powys to be constructed, but the subject can be approached in a slightly different way. Table VIII gives for marriages at each year of age from 20 to 45, the number of women married, the number who bore no children, one, two or three children, within a period of 24 months. In this way prejudiced marriages can probably be ignored, as so small a proportion (2·4 %) of women bear more than one child within this period, that the results are unlikely to be significantly affected by this factor. Unfortunately ages below 20 were not separated in the returns, and can only be given for the group 15—19. If, at the age of 20 and upwards, prejudiced marriages occur to any considerable extent, and if such premarital conceptions are most frequent amongst the more fertile of the group, a disturbing selection may be present in this table; also if fecundity and social class be correlated, and the mean age of marriage in various classes differs, error will be introduced.

Accepting, however, these returns as they stand, the regression line of age at marriage (from 20 upwards) on percentage of women who have borne a child during

this 24 months period is closely linear; a cubic parabola has also been fitted to the data and approximates to the best fitting straight line.

The group 15—19 indicates a falling off in fecundity, and in this corresponds to figures similarly obtained from English and Scotch data kindly sent me by Professor Pearson. The English figures give highest rate at age 18 and the Scotch at age 19, a rapid fall occurring from these ages to that of marriage at 15 years. The sterility rate has been adopted, that is, the percentage of women who did not bear a child during this period of 24 months is shewn in the chart on the bottom right-hand corner of p. 353. It does not seem probable that either of the possibly disturbing factors mentioned above would entirely mask the true trend of the events or result in a regression so nearly linear.

Two other curves are shewn on this chart, p. 353; the first of these gives the percentage of recorded zero families, for marriage at each year from 15 to 27, and thereafter for every second year. This distribution will represent sterility from whatever cause it may arise, either in the husband or the wife. The distribution was fitted by a cubic parabola to the logarithms of the rates per 10,000.

The remaining curve on this chart is that obtained from what is called earlier in this paper the "theoretical zero group," and it is suggested that these groups represent the sterility due only to the female, and form an integral portion of the fertility curves. These three curves form the converse of the fertility curves. The highest mean family is produced at the earliest recorded ages, and the younger the age at marriage within this range the lower the rate of sterility. The purely female sterility curve rises slowly in the earlier ages at marriage and differs in this respect from the almost linear regression of fertility on age at marriage.

The results of the analysis of these figures are, I think, somewhat unexpected, and either, it is true, that in a comparatively homogeneous population, mainly of British origin, the fertility of marriage is directly related to the age at marriage, and steadily diminishes with increasing age at marriage from 15 years upwards, or the groups recorded are not random samples of the women at these ages. If the latter be the case then the selection must be of sufficient intensity to mask completely the true trend of the events. While this is possible, it does not seem probable, having regard to the character of the regression curves obtained.

If, on the other hand, it be true for the race in general that the earlier the marriage the greater the fertility of the wife, then the disadvantage of a rising mean age of marriage in the socially fitter sections of the popula[illegible] evident. Moreover, class fertility rates will have little biological meaning unless the mean age at marriage for the class be taken into consideration. Accepting for the moment the above figures at their face value, a class with a mean age at marriage of 17 would have a mean family per married woman of nearly 8 children, as against just under 5 for a class with a mean age at marriage of 25 years, a fall of 38 per cent. in the fertility due only to postponing the age of marriage.

From a physiological point of view, the decreasing fertility with advancing age

at marriage is of interest. As is shewn in Table VII the younger women maintain their superiority when the period considered is fifteen years of married life for each group, and also to a modified extent in Table VIII when the limit is under 24 months. One is tempted to speculate as to whether early pregnancy has some beneficial effect from this point of view on the female constitution, or whether the male secretion contains some hormone which is essential to the full development of the female reproductive powers. Professor A. Thomson, in a different connection, writes:

> If this be so, then there is absolute proof that the ejaculate contains other ingredients than those alone concerned with fertilization. May there not be some hormone or endoctrine secretion, call it what you will, which by rapid absorption through the tissues of the female, sets agoing, through the agency of the thyroid, the complex mechanism involved in the elaborate preparation of the sexual system to meet its reproductive obligations?

Possibly the earlier such a process is started the more complete its effect.

Dr John W. Harris, in the John Hopkins Hospital *Bulletin*, January 1922, from a study of 160 confinements in white young primiparæ and 340 young coloured primiparæ, of ages from 12 to 16 years, concludes as follows:

> Based upon the study of 500 patients comprised in this report, it seems permissible to conclude that pregnancy and labor are attended by no greater danger to the young primipara than in older women. On the other hand the duration of labor is actually shorter. As our figures show that the size of the children is not inferior to that noted in older women, and that abnormal pelves occur quite as frequently, this result must be attributed to the greater elasticity of the soft parts. Consequently, speaking from a purely obstetrical point of view, the ages under consideration appear to be the optimum time for the occurrence of the first labor.

SUMMARY.

1. That if the fertility data here discussed be reasonably homogeneous, it is probable that in the European population of New Zealand over the age of 15, the younger the wife at marriage the larger will be the mean family of children born alive, unless girls who marry at the earlier ages are drawn from a super-fertile section of the population.

2. That a similar observation is true of the family which survives to adult age.

3. That within the same range of age, namely, 15 and upwards, the percentage of women who bear no children is smaller the younger the age of the wife at marriage.

4. That if fecundity be measured by the percentage of women who bear children within a short period after marriage (24 months) the age of maximum fecundity for New Zealand wives lies probably between the ages 15 to 20.

TABLE I.

Children in Family.

Age at Marriage.	*O*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*N*
15	10	4	8	9	12	19	23	17	20	30	27	31	26	25	11	14	7	5	2	—	—	—	290
17	38	17	41	42	72	99	128	126	125	140	117	112	109	93	63	53	24	13	2	1	1	2	1380
20	175	70	175	258	393	421	465	462	413	428	381	336	237	155	108	70	25	14	3	4	3	—	4421
25	300	100	209	393	433	474	438	354	310	212	165	96	58	31	14	8	4	1	—	1	—	—	3302
30	200	40	174	269	277	266	190	123	69	37	20	10	4	4	1	—	1	—	—	—	—	—	1485
35	180	44	142	155	99	57	24	13	6	3	2	1	1	—	—	—	—	—	—	—	—	—	547
37	195	72	138	106	72	35	18	5	2	1	1	—	—	—	—	—	—	—	—	—	—	—	450
40	223	100	119	59	21	8	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	311

Table giving distribution of families for completed family histories, i.e. women who are 45 years of age and over. Distributions from which constants of Table II were obtained. First "*O*" group records actual observation, second "*O*" group number used in calculation. Tail of array at 37, and 40 also modified. At age 37 observations were as above except that there were 4 cases of 8 children and 3 of 9 and 10. At age 40 after group 3 the numbers of cases of families of 4, 5, etc. were recorded as 13, 7, 6, 4, 4, 3, 3, 0, 1, 1, the last being therefore in group 13.

TABLE II.

Constants of Distributions of Table I.

Age at Marriage	Mean with obs. *O* group	Mean from adjusted *O* group	Mode	σ	β_1	β_2
15	8·2162	8·3862 ± ·1501	—	3·8132 ± ·1068	·0117	2·3671
17	7·7485	7·8667 ± ·0672	—	3·7052 ± ·0476	·0105	2·4544
20	6·5075	6·6469 ± ·0346	5·8387	3·4275 ± ·0246	·0835	2·5841
25	4·4888	4·9116 ± ·0329	3·8686	2·8031 ± ·0233	·3220	3·1197
30	3·3277	3·6862 ± ·0375	2·7828	2·1433 ± ·0265	·7107	4·1551
35	1·8430	2·3400 ± ·0479	1·3903	1·6597 ± ·0338	1·7493	5·9451
37	1·4802	1·9044 ± ·0477	1·0405	1·4988 ± ·0337	1·0376	4·3427
40	0·8901	1·1350 ± ·0424	0·3054	1·1087 ± ·0300	1·4158	4·7273
**20*	—	7·4294 ± ·0525	—	3·4521 ± ·0371	·0044	2·5347

	P	Per cent. of *O* families as recorded	Per cent. of *O* families as obt. from equation to curve	Equation to Curve
15	·89	3·39	1·03	$y = 27{\cdot}07\left(1+\frac{x^2}{109{\cdot}8791}\right)^{2{\cdot}2399}$
17	·64	2·71	1·23	$y = 135{\cdot}8\left(1+\frac{x^2}{123{\cdot}6832}\right)^{2{\cdot}0988}$
20	·10	3·87	1·57	$y = 479{\cdot}9\left(1+\frac{x}{7{\cdot}4085}\right)^{2{\cdot}1737}\left(1-\frac{x}{14{\cdot}1347}\right)^{4{\cdot}1472}$
25	·91	8·56	1·47	$y = 470{\cdot}6\left(1+\frac{x}{5{\cdot}2737}\right)^{2{\cdot}6027}\left(1-\frac{x}{20{\cdot}7528}\right)^{10{\cdot}2124}$
30	·97	12·16	2·75	$y = 299{\cdot}0\, e^{-1{\cdot}1099x}\left(1+\frac{x}{4{\cdot}1814}\right)^{4{\cdot}6284}$
35	·98	26·16	7·95	$y = y_0\,(x-24{\cdot}8342)^{2{\cdot}4489}\, x^{-34{\cdot}3023}$
37	·97	33·56	17·42	$y = 133{\cdot}4\left(1+\frac{x}{1{\cdot}7352}\right)^{1{\cdot}6762}\left(1-\frac{x}{29{\cdot}6928}\right)^{28{\cdot}6837}$
40	·83	53·38	37·90	$y = 138{\cdot}0\left(1+\frac{x}{0{\cdot}7442}\right)^{0{\cdot}7202}\left(1-\frac{x}{15{\cdot}2652}\right)^{14{\cdot}7734}$
**20*	·20	3·93	1·22	$y = 211{\cdot}4\left(1+\frac{x^2}{129{\cdot}8461}\right)^{3{\cdot}9480}$

* Subgroup of group 20 consisting of women over 54 years of age in 1916.

TABLE III.

Women aged 20 *at Marriage.*

Number of living children born.

Age of Mother at date of Census.	Number of Marriages	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	Mean
45	319	18	21	22	36	43	39	38	26	25	13	11	15	9	1	1	1	—	—	—	—	5·30
46	308	13	21	23	37	40	29	32	32	25	17	12	10	3	7	2	2	—	2	1	—	5·59
47	219	10	7	17	29	25	29	23	21	15	8	10	15	3	4	1	1	—	—	—	1	5·72
48	341	7	23	31	44	41	40	34	28	23	29	17	8	6	4	4	1	1	—	—	—	5·56
49	227	2	10	16	23	14	22	27	22	23	25	18	10	8	3	2	1	—	—	—	1	6·56
50	386	20	15	34	45	42	43	50	39	24	20	14	19	10	7	2	1	—	—	1	—	5·61
51	190	4	8	7	14	19	30	18	14	27	14	14	11	2	2	4	1	1	—	—	—	6·51
52	276	13	13	11	25	26	30	28	26	28	26	17	13	8	7	4	—	1	—	—	—	6·31
53	245	11	7	18	23	20	19	30	19	27	18	18	19	9	3	3	1	—	—	—	—	6·41
54	242	8	10	12	24	15	31	18	21	20	24	18	13	13	6	6	3	—	—	—	—	6·77
55	202	3	9	9	14	25	24	24	22	21	20	11	7	6	4	3	—	—	—	—	—	6·37
56	212	5	6	9	15	17	22	25	13	20	22	18	12	11	10	5	1	—	—	1	—	7·18
57	178	6	2	4	10	19	20	12	18	18	23	17	11	4	4	4	2	3	—	—	1	7·35
58	197	9	5	5	15	14	22	14	25	24	25	15	8	6	6	3	—	1	—	—	—	6·85
59	149	2	3	7	11	10	13	10	17	14	16	21	12	7	5	—	1	—	—	—	—	7·35
60	277	29	5	18	14	15	22	26	22	23	16	37	13	19	7	7	2	2	—	—	—	7·80
61	121	4	—	2	3	7	3	10	14	15	16	15	10	11	5	4	1	1	—	—	—	8·44
62	100	2	4	3	4	8	6	11	5	9	13	19	4	3	7	1	—	1	—	—	—	7·66
63	129	4	1	5	4	9	8	8	10	17	17	14	14	5	4	4	3	—	1	1	—	8·12
64	102	2	3	2	2	7	6	12	7	17	11	9	2	6	7	7	1	1	—	—	—	8·14
65	106	3	2	3	1	5	7	12	12	13	8	11	11	6	5	3	2	2	—	—	—	8·21
	4526	175	175	258	393	421	465	462	413	428	381	336	237	155	108	70	25	14	3	4	3	6·51

TABLE IV.

Distribution of Families. Mothers over 44 Years of Age at date of Census

Children living at Census Year.

Age at Marriage	*O*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*N*
17	48	25	56	57	103	132	174	144	137	153	113	108	67	52	31	18	3	1	—	1374
20	215	80	234	353	474	536	536	532	454	393	305	217	137	70	34	21	2	1	4	4383
25	331	120	274	455	510	510	462	348	256	144	105	63	26	10	3	1	1	—	—	3288

TABLE V.

Constants of Distributions in Table IV.

Age at Marriage	Mean adjusted *O* group	Mode	σ	β_1	β_2
17	6·6215 ± ·0589	6·2117	3·2334 ± ·0416	0·0214	2·4550
20	5·6610 ± ·0303	4·8646	2·9739 ± ·0214	0·1266	2·7106
25	4·3020 ± ·0291	3·3868	2·4755 ± ·0206	0·2949	3·0414

	P	Equation to Curve
17	·25	$y=154{\cdot}7\left(1+\frac{x}{7{\cdot}9084}\right)^{2{\cdot}2776}\left(1-\frac{x}{11{\cdot}0256}\right)^{3{\cdot}1753}$
20	·40	$y=559{\cdot}3\left(1+\frac{x}{6{\cdot}3704}\right)^{2{\cdot}4436}\left(1-\frac{x}{14{\cdot}2676}\right)^{5{\cdot}4727}$
25	·99	$y=525{\cdot}7\left(1+\frac{x}{4{\cdot}4965}\right)^{2{\cdot}4050}\left(1-\frac{x}{16{\cdot}5532}\right)^{8{\cdot}6011}$

TABLE VI.

Distribution of Size of Family. Duration of Marriage for each age group, 15 *years.*

Number of Children born.

Age at Marriage	*O*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*N*
15—20	42	16	76	145	147	172	178	134	94	67	32	17	8	3	—	—	—	1089
21—25	204	44	231	465	490	436	347	262	164	96	54	15	9	2	—	1	1	2617
26—30	170	40	165	219	230	190	146	100	63	24	16	1	1	1	—	—	—	1196

TABLE VII.

Constants of Distribution in Table VI.

Age at Marriage	Mean adjusted *O* group	Mode	σ	β_1	β_2
15–20	4·5592 ± ·0478	3·8767	2·3383 ± ·0338	0·1541	2·7591
21—25	3·9687 ± ·0285	2·9290	2·1607 ± ·0201	0·4311	3·3318
26—30	3·5059 ± ·0398	2·6405	2·0421 ± ·0281	0·3229	2·9908

	P	Equation to Curve
15—20	·83	$y = 177{\cdot}6\left(1+\frac{x}{4{\cdot}8294}\right)^{2{\cdot}3831}\left(1-\frac{x}{11{\cdot}7945}\right)^{5{\cdot}8201}$
21—25	·33	$y = 494{\cdot}0\left(1+\frac{x}{3{\cdot}0991}\right)^{1{\cdot}5461}\left(1-\frac{x}{14{\cdot}1039}\right)^{7{\cdot}0302}$
26—30	·92	$y = 230{\cdot}9\left(1+\frac{x}{3{\cdot}2841}\right)^{1{\cdot}7419}\left(1-\frac{x}{12{\cdot}0571}\right)^{6{\cdot}3050}$

TABLE VIII.

Women Married One but under Two Years.

Age at Marriage	Number of Marriages	Number of children				% Marriages with no children	Mean Family
		0	*1*	*2*	*3*		
15—19	279	70	198	11	—	25·1	0·789
20	313	71	227	15	—	22·7	0·821
21	376	102	265	9	—	27·1	0·753
22	603	207	379	16	1	34·3	0·687
23	654	244	391	19	—	37·3	0·656
24	700	292	398	10	—	41·7	0·597
25	689	282	391	16	—	40·9	0·614
26	629	288	321	20	—	45·8	0·574
27	514	214	284	16	—	41·6	0·615
28	472	208	254	10	—	44·1	0·581
29	452	237	210	5	—	52·4	0·487
30	311	151	150	9	1	48·6	0·550
31	247	128	115	4	—	51·8	0·498
32	222	120	100	2	—	54·1	0·468
33	197	107	84	5	1	54·4	0·492
34	123	55	66	2	—	44·7	0·569
35	120	68	51	1	—	56·6	0·442
36	106	63	43	—	—	59·4	0·406
37	83	51	31	1	—	61·4	0·398
38	68	49	16	3	—	72·1	0·324
39	64	49	15	—	—	76·6	0·234
40	60	42	17	1	—	70·0	0·317
41	45	37	8	—	—	82·2	0·178
42	43	35	7	—	1	81·4	0·233
43	19	15	3	—	1	79·0	0·316
44	22	18	4	—	—	81·8	0·182
45	34	32	1	1	—	94·2	0·088
46	29	25	4	—	—	86·3	0·138
47	15	12	3	—	—	80·0	0·200
48	15	13	2	—	—	86·6	0·133
49	14	13	1	—	—	93·9	0·071
50	12	12	—	—	—	100·0	0·000

TABLE IX.

Women aged 20 *at Marriage.*

Number of Children living at Census 1916.

Age of Mother at date of Census.	Number of Marriages	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	Mean
45	319	21	23	33	44	45	34	45	25	20	11	5	7	6	—	—	—	—	—	4·621
46	308	18	28	23	43	44	33	33	27	21	10	14	7	4	1	1	—	—	1	4·864
47	219	11	11	23	31	30	30	23	19	14	6	9	6	1	4	1	—	—	—	4·982
48	341	13	26	38	54	35	56	33	27	18	16	14	3	5	3	—	—	—	—	4·783
49	226	4	14	23	24	18	26	28	22	23	20	10	5	5	3	—	—	1	—	5·708
50	385	26	14	50	48	50	57	39	30	25	18	12	7	5	1	2	—	—	1	4·842
51	190	7	10	10	18	25	35	15	18	18	14	9	6	3	—	2	—	—	—	5·605
52	276	15	17	16	32	29	34	36	25	23	22	9	8	7	2	—	1	—	—	5·460
53	243	12	11	21	27	28	26	27	22	25	20	10	8	4	1	1	—	—	—	5·494
54	242	11	13	19	26	29	25	26	25	16	18	18	9	3	2	2	—	—	—	5·620
55	201	5	9	17	20	31	22	31	23	15	10	9	5	2	2	—	—	—	—	5·418
56	211	6	10	7	23	31	27	19	22	24	14	11	8	5	2	1	—	—	1	5·938
57	178	7	3	8	16	29	13	21	29	19	7	6	9	3	3	4	—	—	1	6·174
58	196	8	6	11	16	23	22	31	27	20	13	13	3	1	1	1	—	—	—	5·776
59	149	2	7	11	12	7	18	17	21	17	14	13	7	3	—	—	—	—	—	6·289
60	277	33	16	16	12	29	25	31	27	19	25	25	11	4	1	3	—	—	—	5·599
61	122	5	—	4	6	11	11	15	14	19	15	5	10	4	2	1	—	—	—	6·951
62	100	2	7	3	8	12	10	11	15	12	12	4	2	1	1	—	—	—	—	5·970
63	128	4	3	6	8	11	8	16	16	17	21	10	2	1	2	2	1	—	—	6·711
64	101	2	3	7	3	9	14	13	12	15	5	5	9	2	2	—	—	—	—	6·515
65	106	3	3	7	3	10	10	22	8	13	14	6	5	1	1	—	—	—	—	6·358
	4518	215	234	353	474	536	536	532	454	393	305	217	137	70	34	21	2	1	4	

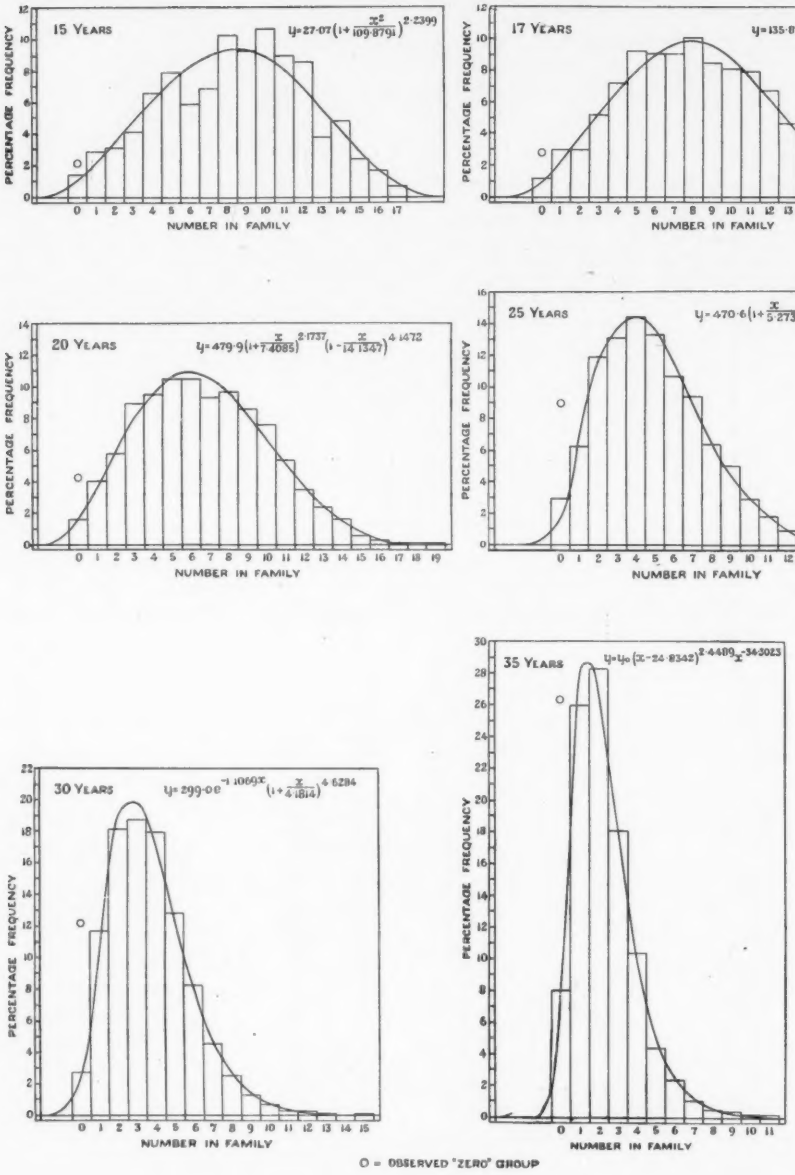
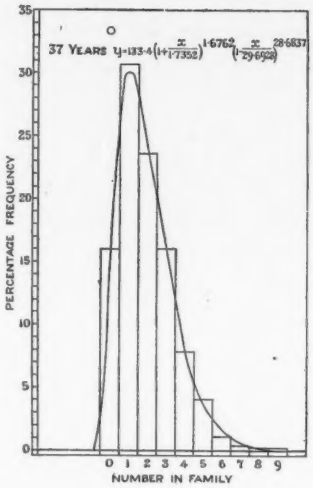
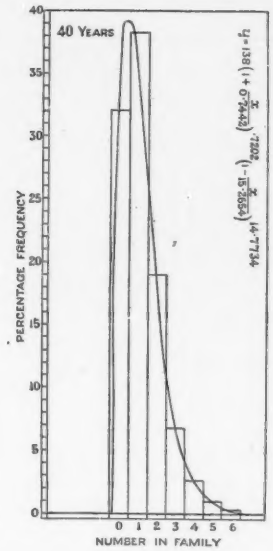
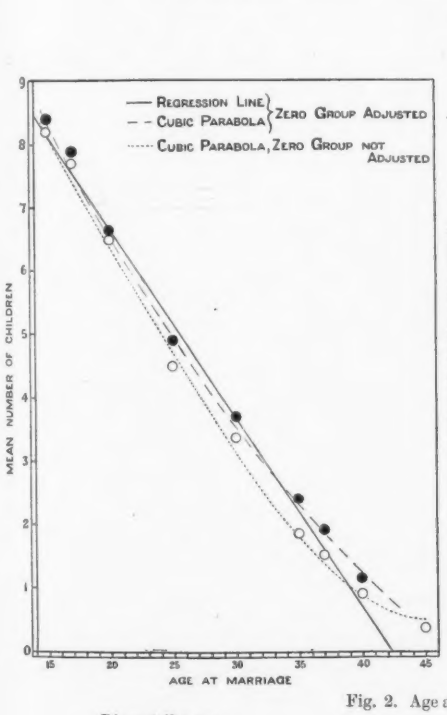


Fig. 1. Age at Marriage of Wife.







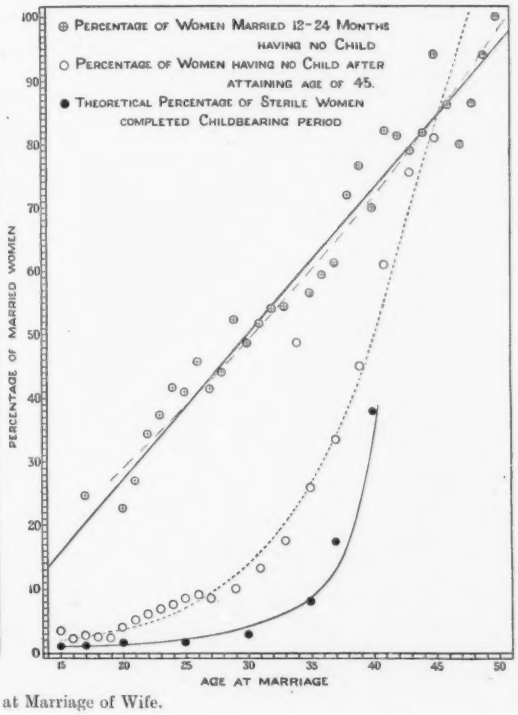

PERCENTAGE OF WOMEN MARRIED 12-24 MONTHS HAVING NO CHILD
PERCENTAGE OF WOMEN HAVING NO CHILD AFTER ATTAINING AGE OF 45.
THEORETICAL PERCENTAGE OF STERILE WOMEN COMPLETED CHILDBEARING PERIOD
PERCENTAGE OF MARRIED WOMEN
AGE AT MARRIAGE

Fig. 2. Age at Marriage of Wife.

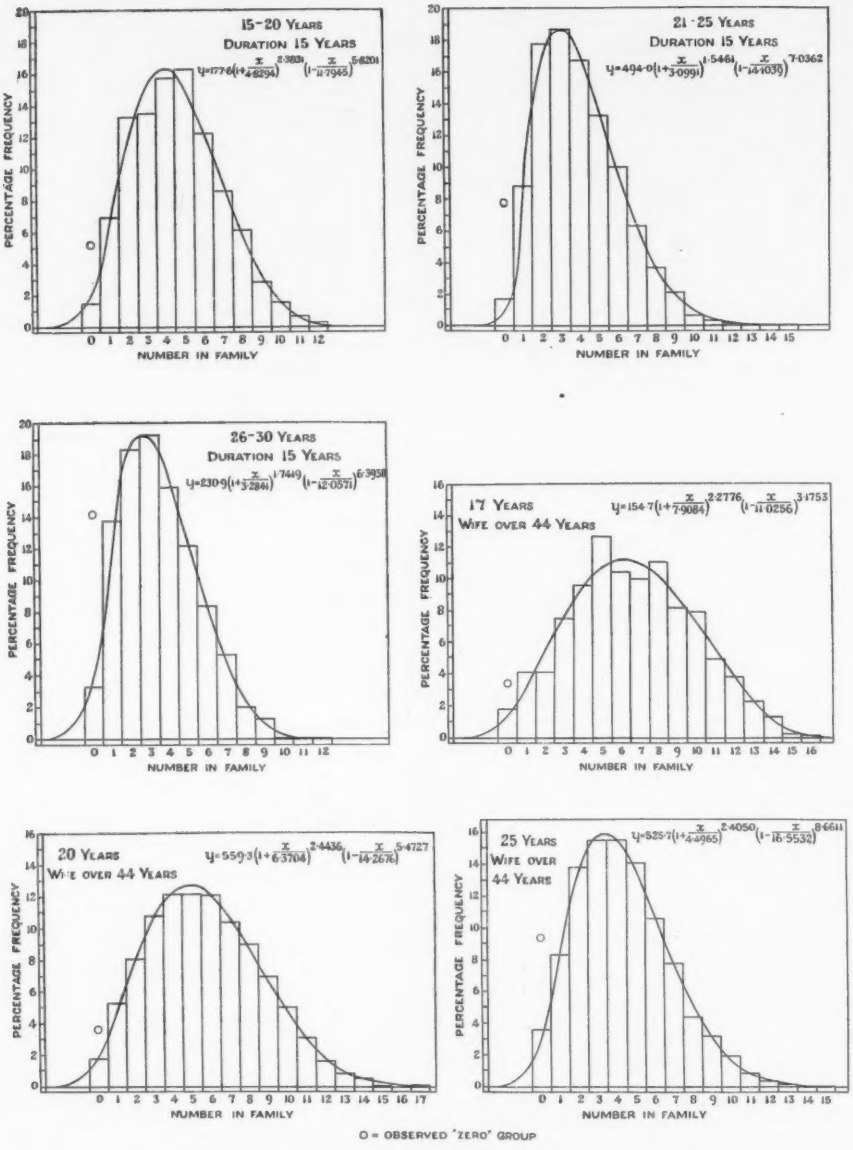


Fig. 3. Age at Marriage of Wife.

ON A CERTAIN SKEW CORRELATION SURFACE.

By E. C. RHODES.

Professor Karl Pearson has discussed in the *Philosophical Transactions of the Royal Society*, and in *Biometrika*, the development of a series of Frequency Curves from a fundamental differential equation which was obtained by considering a certain problem in probability. I have endeavoured, as others have, to obtain from another problem in probability, two general fundamental differential equations, from which to develop a family of Frequency Surfaces, which would extend the Normal Frequency Surface as Professor Pearson's Curves extend the Normal Curve. The two differential equations have been obtained, but so far they have defied general integration. They are of the form

$$\frac{1}{z}\frac{dz}{dx} = \frac{\text{Cubic in } x, y}{\text{Quartic in } x, y},$$

$$\frac{1}{z}\frac{dz}{dy} = \frac{\text{Another Cubic in } x, y}{\text{Same Quartic in } x, y},$$

where x, y are the independent variables and z is the dependent variable or $z\delta x\delta y$ the frequency between x and $x+\delta x$, y and $y+\delta y$.

Particular simple cases were then considered where the denominator included only terms such as

$$c_0 + c_1 x + c_2 y; \quad c_0 + c_1 x + c_2 y + c_3 xy,$$

and so on, but these did not lead to surfaces which were of any real value; the form of these surfaces, however, indicated that just as the general Type I Curve of Professor Pearson represents a frequency distribution of limited range, so the types of surface we wish to consider will be limited to represent frequency for a certain restricted area of the (xy) plane.

Further when I imposed conditions on the differential equations to make the surface unimodal, I obtained a surface which was of little value. Now if we do not impose such conditions, although we find that the surface has more than one mode, yet actually we can arrange that the surface has only one mode as far as its form for the restricted part of the (xy) plane is concerned; the other modes which occur in that part of the (xy) plane, where the ordinate to the surface is either negative or imaginary, we are not interested in.

The numerical illustration I had before me was a frequency distribution of the Barometric Heights at Southampton and Laudale. These data had been used by Professor Pearson and Dr A. Lee in a paper in *Phil. Trans.* A. Vol. 190 (1897), p. 423. This material was definitely skew and seemed a good example of skew correlation.

A glance at Table I will shew that roughly the frequency is contained between two straight lines. I decided therefore to experiment with the surface

$$z = z_0 e^{-lx-my}(1 - x/a + y/b)^p (1 + x/a' - y/b')^{p'};$$

the two straight lines

$$1 - x/a + y/b = 0,$$
$$1 + x/a' - y/b' = 0,$$

being boundaries to the surface, and the exponential term being introduced to ensure that, for x, y large, z should tend to zero.

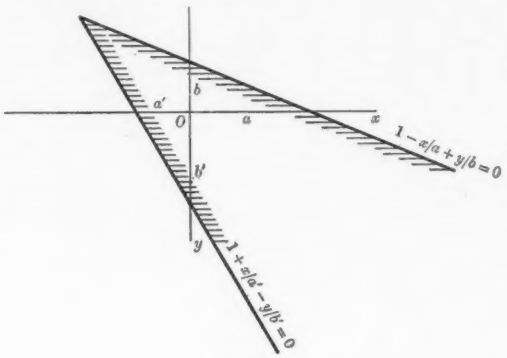


Fig. 1.

Frequency exists in the shaded region.

The equation to this surface may seem artificial, but it is the reasonable type of surface to consider, once the fact that frequency exists only for certain values of x and y is realised, and that in the particular material considered the boundary to the frequency in the (xy) plane consists approximately of two straight lines.

A further point of interest to note is the fact that no effort has been made to use a surface whose arrays shall be Pearson Curves. This question will be referred to later.

Discussion of the surface

$$z = z_0 e^{-lx-my}(1 - x/a + y/b)^p (1 + x/a' - y/b')^{p'}.$$

1. *The Mode.* Differentiating with regard to x and y we have

$$\frac{1}{z}\frac{dz}{dx} = -l - \frac{p/a}{1 - x/a + y/b} + \frac{p'/a'}{1 + x/a' - y/b'} \quad \ldots\ldots\ldots\ldots(1),$$

$$\frac{1}{z}\frac{dz}{dy} = -m + \frac{p/b}{1 - x/a + y/b} - \frac{p'/b'}{1 + x/a' - y/b'} \quad \ldots\ldots\ldots\ldots(2).$$

The mode is given by

$$-\frac{p/a}{L} + \frac{p'/a'}{M} - l = 0,$$

$$+\frac{p/b}{L} - \frac{p'/b'}{M} - m = 0 \text{ where } L \equiv 1 - x/a + y/b,$$
$$M \equiv 1 + x/a' - y/b',$$

i.e by $$\frac{\frac{1}{L}}{-mp'/a'-lp'/b'}=\frac{\frac{1}{M}}{-lp/b-mp/a}=\frac{1}{pp'(1/ab'-1/a'b)},$$

i.e. by $$1-\frac{x}{a}+\frac{y}{b}=\frac{pp'(1/ab'-1/a'b)}{-(mp'/a'+lp'/b')} \quad \text{.....(3),}$$

$$1+\frac{x}{a'}-\frac{y}{b'}=\frac{pp'(1/ab'-1/a'b)}{-(lp/b+mp/a)} \quad \text{.....(4).}$$

There is therefore one mode to this surface, which is at the origin if

$$\left.\begin{aligned} l&=-p/a+p'/a' \\ m&=\ \ p/b-p'/b' \end{aligned}\right\} \quad \text{.....(5).}$$

We will assume that these conditions are satisfied and that the mode of the surface is the origin.

2. *Discussion of the arrays of the surface.*

Equations (1) and (2) give us easily

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b'}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a'}=-\frac{l}{b'}-\frac{m}{a'}-\left(\frac{p/ab'-p/a'b}{1-x/a+y/b}\right),$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a}=-\frac{l}{b}-\frac{m}{a}+\left(\frac{p'/a'b-p'/ab'}{1+x/a'-y/b'}\right).$$

But (3) and (4) shew us that when the origin is the mode,

$$l/b'+m/a'=-p(1/ab'-1/a'b),$$

$$l/b+m/a=-p'(1/ab'-1/a'b).$$

We have therefore, calling

$$\frac{1}{a'b}-\frac{1}{ab'}=X,$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b'}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a'}=-pX+\frac{pX}{1-x/a+y/b}=\frac{pX(x/a-y/b)}{1-x/a+y/b} \quad \text{.....(6),}$$

$$\frac{1}{z}\frac{dz}{dx}\frac{1}{b}+\frac{1}{z}\frac{dz}{dy}\frac{1}{a}=-p'X+\frac{p'X}{1+x/a'-y/b'}=\frac{p'X(-x/a'+y/b')}{1+x/a'-y/b'} \quad \text{.....(7).}$$

Therefore: $$\left(\frac{1}{b'}\frac{dz}{dx}+\frac{1}{a'}\frac{dz}{dy}\right)(1-x/a+y/b)=pXz(x/a-y/b).$$

Let us integrate this equation throughout the range of x, say from x_1 to x_2, keeping y constant.

Now $$\int_{x_1}^{x_2}\frac{dz}{dx}dx=[z]_{x_1}^{x_2}=0,$$

since z vanishes at the boundary lines of the surface;

$$\int_{x_1}^{x_2}\frac{dz}{dx}x\,dx=[zx]_{x_1}^{x_2}-\int_{x_1}^{x_2}z\,dx=-z_y,$$

z_y being the area of the array;

$$\int_{x_1}^{x_2} \frac{dz}{dx}\, y\, dx = y\,[z]_{x_1}^{x_2} = 0,$$

$$\int_{x_1}^{x_2} \frac{dz}{dy}\, dx = \frac{d}{dy}\int_{x_1}^{x_2} z\, dx = \frac{d}{dy}(z_y),$$

$$\int_{x_1}^{x_2} \frac{dz}{dy}\, x\, dx = \frac{d}{dy}\int_{x_1}^{x_2} z\, x\, dx = \frac{d}{dy}(z_y \bar{x}_y),$$

$\bar{x}_y$ being the mean of the array;

$$\int_{x_1}^{x_2} \frac{dz}{dy}\, y\, dx = y\,\frac{d}{dy}\int_{x_1}^{x_2} z\, dx = y\,\frac{d}{dy}(z_y).$$

We have then

$$\frac{1}{b'}\left(+\frac{1}{a}z_y\right) + \frac{1}{a'}\left(\frac{d}{dy}(z_y) - \frac{1}{a}\frac{d}{dy}(z_y\bar{x}_y) + \frac{y}{b}\frac{d}{dy}(z_y)\right) = pX\left(\frac{1}{a}\bar{x}_y - \frac{1}{b}y\right)z_y,$$

i.e.
$$\frac{1}{a'z_y}\left(\frac{dz_y}{dy}\right)\left[1 - \frac{\bar{x}_y}{a} + \frac{y}{b}\right] + \frac{1}{ab'} = pX\left(\frac{\bar{x}_y}{a} - \frac{y}{b}\right) + \frac{1}{aa'}\frac{d}{dy}(\bar{x}_y) \quad \text{.....(8).}$$

Similarly equation (7) gives us

$$\frac{1}{az_y}\left(\frac{dz_y}{dy}\right)\left[1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'}\right] - \frac{1}{a'b} = p'X\left(-\frac{\bar{x}_y}{a'} + \frac{y}{b'}\right) - \frac{1}{aa'}\frac{d}{dy}(\bar{x}_y) \quad \text{.....(9).}$$

Adding these two equations we obtain

$$\frac{1}{z_y}\frac{dz_y}{dy}\left(\frac{1}{a} + \frac{1}{a'} + yX\right) = X\left(p\left(\frac{\bar{x}_y}{a} - \frac{y}{b}\right) - p'\left(\frac{\bar{x}_y}{a'} - \frac{y}{b'}\right) + 1\right).$$

If we call
$$1 - \frac{\bar{x}_y}{a} + \frac{y}{b} = Z,$$

$$1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'} = Y,$$

then
$$\frac{Z}{a'} + \frac{Y}{a} = \frac{1}{a} + \frac{1}{a'} + Xy = u \text{ (say)},$$

and the equation above becomes

$$\begin{aligned}\frac{1}{z_y}\frac{dz_y}{dy}.\,u &= X\left[p(1-Z) - p'(Y-1) + 1\right] \\ &= X\left[p + p' + 1 - p'Y - p\left(u - \frac{Y}{a}\right)a'\right] \\ &= X(R - 1 - pua' - a'Yl) \quad \text{.....(10),}\end{aligned}$$

where $p + p' + 2 = R$.

Also we can write equation (9) in the form

$$\begin{aligned}\frac{1}{az_y}\frac{dz_y}{dy}.\,Y &= \frac{1}{a'b} + p'X(1-Y) - \frac{1}{a}\left(\frac{dY}{dy} + \frac{1}{b'}\right) \\ &= X\left(p' + 1 - p'Y - \frac{1}{a}\frac{dY}{X\,dy}\right).\end{aligned}$$

But we have $Xdy = du$; we can therefore write this last equation

$$\frac{1}{a}\frac{Y}{z_y}\frac{dz_y}{du} = p' + 1 - p'Y - \frac{1}{a}\frac{dY}{du} \quad \text{.....(11).}$$

Now if we call

$$\frac{1}{z_y}\frac{dz_y}{du} + pa' = \frac{dv}{du} \quad \text{.....(12)}$$

we can write equation (10):

$$\frac{u}{z_y}\frac{dz_y}{du} = R - 1 - pa'u - a'lY,$$

as

$$u\frac{dv}{du} = R - 1 - a'lY \quad \text{.....(13),}$$

and equation (11) as

$$\frac{Y}{a}\left(\frac{dv}{du} - pa'\right) = p' + 1 - p'Y - \frac{1}{a}\frac{dY}{du},$$

i.e.

$$\frac{Y}{a}\frac{dv}{du} = p' + 1 + a'Y\left(-\frac{p'}{a'} + \frac{p}{a}\right) - \frac{1}{a}\frac{dY}{du}$$

$$= p' + 1 - a'lY - \frac{1}{a}\frac{dY}{du} \quad \text{.....(14).}$$

Write

$$\frac{dv}{du} = \frac{1}{w}\frac{dw}{du} \quad \text{.....(15).}$$

Differentiate (13): we get

$$\frac{d}{du}\left(u\frac{dw}{du}\right) = \frac{d}{du}\left((R-1)w - a'lYw\right),$$

i.e.

$$u\frac{d^2w}{du^2} + \frac{dw}{du} = (R-1)\frac{dw}{du} - a'lY\frac{dw}{du} - a'lw\frac{dY}{du}$$

$$= (R-1)\frac{dw}{du} - a'lY\frac{dw}{du} - a'la\left[(p'+1)w - a'lYw - \frac{Y}{a}\frac{dw}{du}\right] \quad \text{from (14)}$$

$$= (R-1)\frac{dw}{du} - aa'lw(p' + 1 - a'lY)$$

$$= (R-1)\frac{dw}{du} - aa'lw\left(p' + 1 + \frac{u}{w}\frac{dw}{du} - R + 1\right), \quad \text{from (13)}$$

i.e.

$$u\frac{d^2w}{dw^2} = \frac{dw}{du}(R - 2 - aa'lu) + aa'lpw,$$

or

$$u\frac{d^2w}{dw^2} + \frac{dw}{du}(aa'lu - (R-2)) - aa'lpw = 0 \quad \text{.....(16).}$$

This is the differential equation to obtain w which, from (15) and (12), is given by

$$\log w = \text{const.} + \log z_y + pa'u,$$

or

$$w = z_y e^{pa'u} \times \text{const.}$$

Thus from (16) we can obtain the total of the x-array—in other words this is the differential equation giving the y-margin curve.

The solution to this equation may be obtained in a series of powers of u in the usual way. We obtain

$$w = C\left(u^{R-1} - \frac{aa'ls'}{R}u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}u^{R+1} - \frac{(aa'l)^3}{3!}\frac{s'.s'+1.s'+2}{R.R+1.R+2}u^{R+2} + \ldots\right) \quad \ldots\ldots\ldots(17),$$

or the curve to the y-margin is

$$z_y = Ce^{-pa'\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)}\left[\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)^{R-1} - \frac{aa'ls'}{R}\left(\frac{1}{a}+\frac{1}{a'}+Xy\right)^R + \ldots\right],$$

putting in the expression for u, where $s' = p'+1$.

From (13) we have $$a'lY = R - 1 - \frac{u}{w}\frac{dw}{du};$$

and since $$Y = 1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'},$$

this enables us to obtain the mean ($\bar{x}_y$) of an array for any y; in other words this equation is that to the regression line of x on y.

Since

$$\frac{dw}{du} = C\left((R-1)u^{R-2} - \frac{aa'ls'}{R}.R.u^{R-1} + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}(R+1)u^R - \ldots\right)$$

$$= C\frac{(R-1)}{u}\left(u^{R-1} - \frac{aa'l}{R-1}.s'.u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R-1.R}.u^{R+1} - \ldots\right),$$

it follows that

$$a'lY = (R-1)\left[1 - \frac{u^{R-1} - \frac{aa'l}{R-1}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R-1.R}u^{R+1} - \ldots}{u^{R-1} - \frac{aa'l}{R}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}u^{R+1} - \ldots}\right]$$

$$= \frac{aa'ls'}{R}\frac{\left[u^R - aa'l\frac{s'+1}{R+1}u^{R+1} + \frac{(aa'l)^2}{2!}\frac{s'+1.s'+2}{R+1.R+2}u^{R+2} - \ldots\right]}{u^{R-1} - \frac{aa'l}{R}s'u^R + \frac{(aa'l)^2}{2!}\frac{s'.s'+1}{R.R+1}u^{R+1} - \ldots}.$$

If we call $$S_{R.s'} = u^{R-1} - \frac{aa'l}{R}s'u^R + \ldots,$$

$$S_{R+1.s'+1} = u^R - aa'l\frac{(s'+1)}{R+1}u^{R+1} + \ldots,$$

then $$Y = \frac{as'}{R}\frac{S_{R+1.s'+1}}{S_{R.s'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and $$w = CS_{R.s'}.$$

We see that the regression line is an infinite series in u or y.

Referring back to equations (13) and (14), we have

$$\frac{Y}{au} = \frac{p'+1-a'lY-\frac{1}{a}\frac{dY}{du}}{R-1-a'lY},$$

i.e. $$\frac{1}{a}\frac{dY}{du} + a'lY - (p'+1) + ((R-1)-a'lY)\frac{Y}{au} = 0 \quad \ldots\ldots\ldots\ldots(19)$$

is the differential equation for Y, the solution to which is not as simple an expression as that already obtained for Y from w.

But we can note an interesting point here. If the regression is linear, equation (18) shews that

$$Y = \frac{as'}{R}(u + A) \text{ where } A \text{ is some constant.}$$

(19) then gives

$$\frac{1}{a}\frac{as'}{R} + a'l\frac{as'}{R}(u + A) - s' + \left[R - 1 - a'l\frac{as'}{R}(u + A)\right]\frac{s'}{R}\left(1 + \frac{A}{u}\right) = 0.$$

This shews that $A = 0$ and $\frac{s'}{R} = 1$; i.e. $s = p + 1 = 0$, and the regression line is

$$1 + \frac{\bar{x}_y}{a'} - \frac{y}{b'} = a\left(\frac{1}{a} + \frac{1}{a'} + Xy\right),$$

i.e.
$$\frac{\bar{x}_y}{a'} = \frac{a}{a'} + y\left(\frac{a}{a'b} - \frac{1}{b'} + \frac{1}{b'}\right)$$
$$= \frac{a}{a'} + \frac{ay}{a'b},$$

i.e.
$$\frac{\bar{x}_y}{a} = 1 + \frac{y}{b}.$$

(13) gives now
$$u\frac{dv}{du} = R - 1 - a'lau,$$

whence
$$v = \text{const.} + (R - 1)\log u - aa'lu,$$
$$z_y e^{pa'u} = Ce^{-aa'lu}u^{R-1},$$

or the margin is a Type III Curve.

We see that the regression line is $\frac{\bar{x}_y}{a} = 1 + \frac{y}{b}$; i.e. it coincides with one of the bounding lines; the surface has therefore been reduced to a plane, and the correlation is perfect.

Thus we see that this type of surface has only linear regression when the correlation coefficient is unity; so we shall expect that for surfaces of this type the regression will be reasonably linear only when there is high correlation between the two characters. This happens in the case we are considering—the Barometric Height material—for the regression is apparently sensibly linear and the correlation coefficient is + ·78.

We have so far obtained the curve to the y-margin, and the equation to the regression line of x on y. In the same way we may obtain the curve to the x-margin and the regression line of y on x.

3. *The moments of the surface.*

We may write equations (6) and (7)

$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)(1 - x/a + y/b) = pXz\,(x/a - y/b) \quad \ldots\ldots\ldots\ldots(20),$$

$$\left(\frac{1}{b}\frac{dz}{dx} + \frac{1}{a}\frac{dz}{dy}\right)(1 + x/a' - y/b') = p'Xz\,(-x/a' + y/b') \quad \ldots\ldots\ldots(21).$$

Integrate equation (20) throughout the whole range of x and y. We have

$$\iint \frac{dz}{dx}\,dxdy = \int [z]\,dy = 0,$$

$$\iint \frac{dz}{dx}\,x\,dxdy = \int [zx]\,dy - \iint z\,dxdy = -N, \text{ the total frequency,}$$

$$\iint \frac{dz}{dx}\,x^m y^n\,dxdy = \int [zx^m]\,y^n\,dy - \iint z m x^{m-1} y^n\,dxdy = -Nmp'_{m-1,\,n},$$

$$\iint \frac{dz}{dy}\,x^m y^n\,dxdy = \int [zy^n]\,x^m\,dx - \iint z n x^m y^{n-1}\,dxdy = -nNp'_{m,\,n-1},$$

where $p'_{m,n}$ is the m, nth product-moment about an origin not at the mean.

We get from (20)

$$\frac{1}{b'}\left[-\frac{1}{a}(-N)\right] + \frac{1}{a'}\left[\frac{1}{b}(-N)\right] = pX\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right)N,$$

i.e.
$$pX\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right) + X = 0,$$

or
$$p\left(\frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right) + 1 = 0.$$

Similarly from (21)

$$p'\left(-\frac{p'_{10}}{a'} + \frac{p'_{01}}{b'}\right) + 1 = 0,$$

whence
$$\left.\begin{aligned} p'_{10} &= \frac{1}{X}\left(\frac{1}{pb'} + \frac{1}{p'b}\right) \\ p'_{01} &= \frac{1}{X}\left(\frac{1}{pa'} + \frac{1}{p'a}\right) \end{aligned}\right\} \quad \text{.....(21 (i)),}$$

which gives the distance of the mean from the mode.

Change the origin of the surface from the mode to the mean, i.e. put

$$x = x + p'_{10} \text{ and } y = y + p'_{01}.$$

Then (20) and (21) become

$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)\left(1 - \frac{x}{a} + \frac{y}{b} - \frac{p'_{10}}{a} + \frac{p'_{01}}{b}\right) = pXz\left(\frac{x}{a} - \frac{y}{b} + \frac{p'_{10}}{a} - \frac{p'_{01}}{b}\right),$$

i.e.
$$\left(\frac{1}{b'}\frac{dz}{dx} + \frac{1}{a'}\frac{dz}{dy}\right)\left(1 - \frac{x}{a} + \frac{y}{b} + \frac{1}{p}\right) = pXz\left(\frac{x}{a} - \frac{y}{b} - \frac{1}{p}\right),$$

and
$$\left(\frac{1}{b}\frac{dz}{dx} + \frac{1}{a}\frac{dz}{dy}\right)\left(1 + \frac{x}{a'} - \frac{y}{b'} + \frac{1}{p'}\right) = p'Xz\left(-\frac{x}{a'} + \frac{y}{b'} - \frac{1}{p'}\right),$$

whence multiplying these by $x^m y^n$ and integrating we have

$$\frac{1}{b'}\left(\frac{s}{p}\,m p_{m-1,n} - \frac{1}{a}(m+1)\,p_{m,n} + \frac{m}{b}\,p_{m-1,n+1}\right)$$
$$+ \frac{1}{a'}\left(\frac{s}{p}\,n p_{m,n-1} - \frac{1}{a}\,n p_{m+1,n-1} + \frac{1}{b}(n+1)\,p_{m,n}\right) = -pX\left(\frac{p_{m+1,n}}{a} - \frac{p_{m,n+1}}{b} - \frac{p_{m,n}}{p}\right)$$
$$\text{.....(22),}$$

and

$$\frac{1}{b}\left(\frac{s'}{p'}mp_{m-1,n}+\frac{1}{a'}(m+1)\,p_{m,n}-\frac{m}{b'}p_{m-1,n+1}\right)$$
$$+\frac{1}{a}\left(\frac{s'}{p'}np_{m,n-1}+\frac{1}{a'}np_{m+1,n-1}-\frac{1}{b'}(n+1)\,p_{m,n}\right)=-p'X\left(-\frac{p_{m+1,n}}{a'}+\frac{p_{m,n+1}}{b'}-\frac{p_{m,n}}{p'}\right)$$
.....(23),

where $p_{m,n}$ is a product-moment about the mean.

When $m=1$, $n=0$, (22) gives

$$-\frac{1}{b'}\frac{s}{p}=-pX\left(\frac{p_{20}}{a}-\frac{p_{11}}{b}\right) \quad \text{.....(24).}$$

(23) gives
$$\frac{1}{b}\frac{s'}{p'}=-p'X\left(-\frac{p_{20}}{a'}+\frac{p_{11}}{b'}\right) \quad \text{.....(25),}$$

$$\therefore \quad \frac{s}{b'^2p^2}+\frac{s'}{b^2p'^2}=-Xp_{20}\left(\frac{1}{ab'}-\frac{1}{a'b}\right)=X^2p_{20},$$

$$\therefore \quad p_{20}=\frac{1}{X^2}\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right) \quad \text{.....(25 bis),}$$

and
$$p_{02}=\frac{1}{X^2}\left(\frac{s}{p^2a'^2}+\frac{s'}{p'^2a^2}\right) \text{ similarly.}$$

Also
$$\frac{s}{p^2a'b'}+\frac{s'}{p'^2ab}=Xp_{11}\left(\frac{1}{a'b}-\frac{1}{ab'}\right)=X^2p_{11},$$

$$\therefore \quad p_{11}=\frac{1}{X^2}\left(\frac{s}{p^2a'b'}+\frac{s'}{p'^2ab}\right),$$

and we have here equations which give us σ_x, σ_y and r in terms of the constants of the surface.

Note that

$$p_{20}p_{02}-p_{11}^2=\frac{ss'}{X^4}\left(\frac{1}{a^2b'^2}+\frac{1}{a'^2b^2}-\frac{2}{aa'bb'}\right)\frac{1}{p^2p'^2}=\frac{ss'}{p^2p'^2X^2},$$

whence
$$1-r^2=\frac{ss'X^2}{p^2p'^2\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right)\left(\frac{s}{p^2a'^2}+\frac{s'}{p'^2a^2}\right)} \quad \text{.....(26).}$$

When $m=1$, $n=1$, (22) gives

$$\frac{1}{b'}\left(-\frac{2}{a}p_{11}+\frac{1}{b}p_{02}\right)+\frac{1}{a'}\left(-\frac{1}{a}p_{20}+\frac{2}{b}p_{11}\right)=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}-\frac{p_{11}}{p}\right),$$

i.e.
$$\frac{p_{02}}{bb'}-\frac{p_{20}}{aa'}+2p_{11}X=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}\right)+p_{11}X,$$

i.e.
$$\frac{p_{02}}{bb'}-\frac{p_{20}}{aa'}+p_{11}X=-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}\right).$$

(23) gives

$$\frac{1}{b}\left(\frac{2}{a'}p_{11}-\frac{1}{b'}p_{02}\right)+\frac{1}{a}\left(\frac{1}{a'}p_{20}-\frac{2}{b'}p_{11}\right)=-p'X\left(-\frac{p_{21}}{a'}+\frac{p_{12}}{b'}-\frac{p_{11}}{p'}\right),$$

i.e.
$$-\frac{p_{02}}{bb'}+\frac{p_{20}}{aa'}+p_{11}X=-p'X\left(-\frac{p_{21}}{a'}+\frac{p_{12}}{b'}\right).$$

But

$$\frac{p_{02}}{bb'}-\frac{p_{20}}{aa'}+p_{11}\left(\frac{1}{a'b}-\frac{1}{ab'}\right)=\left(\frac{p_{02}}{b}-\frac{p_{11}}{a}\right)\frac{1}{b'}+\frac{1}{a'}\left(\frac{p_{11}}{b}-\frac{p_{20}}{a}\right)=\frac{s}{a'p^2Xb'}+\frac{s}{b'p^2Xa'},$$

and

$$-\frac{p_{02}}{bb'}+\frac{p_{20}}{aa'}+p_{11}X=\frac{1}{b}\left(\frac{p_{11}}{a'}-\frac{p_{02}}{b'}\right)+\frac{1}{a}\left(\frac{p_{20}}{a'}-\frac{p_{11}}{b'}\right)=\frac{s'}{ap'^2Xb}+\frac{s'}{bp'^2Xa}.$$

Therefore

$$-pX\left(\frac{p_{21}}{a}-\frac{p_{12}}{b}\right)=\frac{2s}{Xp^2a'b'},$$

$$-p'X\left(-\frac{p_{21}}{a'}+\frac{p_{12}}{b'}\right)=\frac{2s'}{Xp'^2ab},$$

whence

$$p_{21}X=\frac{2s}{X^2p^3a'b'^2}+\frac{2s'}{X^2p'^3ab^2},$$

$$\therefore\ p_{21}=\frac{2}{X^3}\left(\frac{s}{p^3a'b'^2}+\frac{s'}{p'^3ab^2}\right) \quad \text{.....(27),}$$

and

$$p_{12}=\frac{2}{X^3}\left(\frac{s}{p^3a'^2b'}+\frac{s'}{p'^3a^2b}\right) \quad \text{.....(28).}$$

When $m=2$, $n=0$, (22) gives

$$\frac{1}{b'}\left(-\frac{3}{a}p_{20}+\frac{2}{b}p_{11}\right)+\frac{1}{a'}\left(\frac{1}{b}p_{20}\right)=-pX\left(\frac{p_{30}}{a}-\frac{p_{21}}{b}-\frac{p_{20}}{p}\right),$$

i.e.

$$\frac{1}{b'}\left(-\frac{2}{a}p_{20}+\frac{2}{b}p_{11}\right)=-pX\left(\frac{p_{30}}{a}-\frac{p_{21}}{b}\right),$$

i.e.

$$\frac{1}{b'^2}\frac{2s}{p^2X}=-pX\left(\frac{p_{30}}{a}-\frac{p_{21}}{b}\right) \quad \text{.....(29).}$$

(23) gives

$$\frac{2s'}{b^2p'^2X}=-p'X\left(-\frac{p_{30}}{a'}+\frac{p_{21}}{b'}\right) \quad \text{.....(30).}$$

Therefore

$$p_{30}X=\frac{2s'}{b^2p'^3X^2}+\frac{2s}{b'^2p^2X^2},$$

i.e.

$$p_{30}=\frac{2}{X^3}\left(\frac{s}{p^2b'^3}+\frac{s'}{p'^3b^3}\right) \quad \text{.....(31),}$$

and similarly

$$p_{03}=\frac{2}{X^3}\left(\frac{s}{p^3a'^3}+\frac{s'}{p'^3a^3}\right) \quad \text{.....(32).}$$

When $m=3$, $n=0$, (22) gives

$$\frac{1}{b'}\left(\frac{3s}{p}p_{20}-\frac{4}{a}p_{30}+\frac{3}{b}p_{21}\right)+\frac{1}{a'}\frac{1}{b}p_{30}=-pX\left(\frac{p_{40}}{a}-\frac{p_{31}}{b}-\frac{p_{30}}{p}\right).$$

(23) gives

$$\frac{1}{b}\left(\frac{3s'}{p'}p_{20}+\frac{4}{a'}p_{30}-\frac{3}{b'}p_{21}\right)-\frac{1}{a}\frac{1}{b'}p_{30}=-p'X\left(-\frac{p_{40}}{a'}+\frac{p_{31}}{b'}-\frac{p_{30}}{p'}\right).$$

The first equation is

$$\frac{3s}{pb'}p_{30}-\frac{3}{b'}\left(\frac{p_{30}}{a}-\frac{p_{21}}{b}\right)+p_{30}X=-pX\left(\frac{p_{40}}{a}-\frac{p_{31}}{b}\right)+p_{30}X,$$

i.e. $$\frac{3s}{pb'}p_{20}+\frac{3}{b'}\frac{2s}{p^3X^2b'^2}=-pX\left(\frac{p_{40}}{a}-\frac{p_{31}}{b}\right) \text{ from (29).}$$

The second equation is

$$\frac{3s'}{p'b}p_{20}+\frac{3}{b}\left(\frac{p_{30}}{a'}-\frac{p_{21}}{b'}\right)+p_{30}X=-p'X\left(-\frac{p_{40}}{a'}+\frac{p_{31}}{b'}\right)+p_{30}X,$$

i.e. $$\frac{3s'}{p'b}p_{20}+\frac{3}{b}\frac{2s'}{p'^3X^2b^2}=-p'X\left(-\frac{p_{40}}{a'}+\frac{p_{31}}{b'}\right) \text{ from (30).}$$

Eliminate p_{31} from these equations. We have

$$3p_{20}\left(\frac{s}{p^2b'^2}+\frac{s'}{p'^2b^2}\right)+\frac{6}{X^2}\left(\frac{s}{p^4b'^4}+\frac{s'}{p'^4b^4}\right)=X^2p_{40},$$

$$\therefore\ p_{40}-3p_{20}^2=\frac{6}{X^4}\left(\frac{s}{p^4b'^4}+\frac{s'}{p'^4b^4}\right) \text{ from (25 bis) } \ldots\ldots\ldots\ldots(33).$$

Similarly $$p_{04}-3p_{02}^2=\frac{6}{X^4}\left(\frac{s}{p^4a'^4}+\frac{s'}{p'^4a^4}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Now if we call $$\frac{ap'}{a'p}=\theta;\quad \frac{bp'}{b'p}=\phi;\quad \frac{s'}{s}=\lambda,$$

we have

$$p_{20}=\frac{s}{X^2p'^2b^2}(\phi^2+\lambda),$$

$$p_{02}=\frac{s}{X^2p'^2a^2}(\theta^2+\lambda),$$

$$p_{30}=\frac{2s}{X^3p'^3b^3}(\phi^3+\lambda),$$

$$p_{03}=\frac{2s}{X^3p'^3a^3}(\theta^3+\lambda),$$

$$p_{11}=\frac{s}{X^2p'^2ab}(\theta\phi+\lambda),$$

$$p_{21}=\frac{2s}{X^3p'^3ab^2}(\theta\phi^2+\lambda),$$

$$p_{12}=\frac{2s}{X^3p'^3a^2b}(\theta^2\phi+\lambda),$$

$$p_{40}-3p_{20}^2=\frac{6s}{X^4p'^4b^4}(\phi^4+\lambda),$$

$$p_{04}-3p_{02}^2=\frac{6s}{X^4p'^4a^4}(\theta^4+\lambda).$$

Calling $$\beta_{10}=\frac{p_{30}^2}{p_{20}^3};\quad \beta_{01}=\frac{p_{03}^2}{p_{02}^3};\quad \beta_{20}=\frac{p_{40}}{p_{20}^2};\quad \beta_{02}=\frac{p_{04}}{p_{02}^2};$$

$$q_{21}=\frac{p_{21}}{p_{20}\sqrt{p_{02}}};\quad q_{12}=\frac{p_{12}}{p_{02}\sqrt{p_{20}}};\quad r=\frac{p_{11}}{\sqrt{p_{20}p_{02}}},$$

we have $$r = \frac{(\theta\phi + \lambda)}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}} \quad \text{.....(35),}$$

$$\beta_{10} = \frac{4}{s}\frac{(\phi^3+\lambda)^2}{(\phi^2+\lambda)^3}; \quad \beta_{01} = \frac{4}{s}\frac{(\theta^3+\lambda)^2}{(\theta^2+\lambda)^3} \quad \text{.....(36),}$$

$$q_{21} - r\sqrt{\beta_{10}} = \frac{2(\theta\phi^2+\lambda)}{\sqrt{s}(\phi^2+\lambda)\sqrt{\theta^2+\lambda}} - \frac{2(\theta\phi+\lambda)(\phi^3+\lambda)}{\sqrt{s}(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}$$

$$= \frac{2}{\sqrt{s}}\frac{(\theta\phi^2+\lambda)(\phi^2+\lambda) - (\theta\phi+\lambda)(\phi^3+\lambda)}{(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}$$

$$= \frac{2}{\sqrt{s}}\frac{\lambda(\phi^2+\theta\phi^2-\phi^3-\theta\phi)}{(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}} = \frac{2\lambda}{\sqrt{s}}\frac{\phi(1-\phi)(\phi-\theta)}{(\phi^2+\lambda)^2\sqrt{\theta^2+\lambda}}.$$

Now $$\sqrt{1-r^2} = \frac{(\phi-\theta)\sqrt{\lambda}}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}} \text{ from (26) above,}$$

$$\therefore \frac{q_{21} - r\sqrt{\beta_{10}}}{\sqrt{1-r^2}} = \frac{2\sqrt{\lambda}}{\sqrt{s}}\frac{\phi(1-\phi)}{(\phi^2+\lambda)^{\frac{3}{2}}}.$$

Similarly $$\frac{q_{12} - r\sqrt{\beta_{01}}}{\sqrt{1-r^2}} = \frac{2\sqrt{\lambda}}{\sqrt{s}}\frac{\theta(1-\theta)}{(\theta^2+\lambda)^{\frac{3}{2}}} \quad \text{.....(37).}$$

These functions $q_{21} - r\sqrt{\beta_{10}}$; $q_{12} - r\sqrt{\beta_{01}}$ have entered elsewhere.

Again from (33) and (34):

$$\beta_{20} - 3 = \frac{6}{s}\frac{\phi^4+\lambda}{(\phi^2+\lambda)^2}, \qquad \beta_{02} - 3 = \frac{6}{s}\frac{\theta^4+\lambda}{(\theta^2+\lambda)^2}.$$

Let us consider $$2\beta_{20} - 3\beta_{10} - 6.$$

This is $$\frac{12}{s(\phi^2+\lambda)^3}[(\phi^4+\lambda)(\phi^2+\lambda) - (\phi^3+\lambda)^2]$$

$$= \frac{12\lambda\phi^2(1-\phi)^2}{s(\phi^2+\lambda)^3},$$

$$\therefore \frac{q_{21} - r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}\sqrt{3} = \sqrt{2\beta_{20} - 3\beta_{10} - 6}.$$

Similarly $$\frac{q_{12} - r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}\sqrt{3} = \sqrt{2\beta_{02} - 3\beta_{01} - 6}.$$

These relations therefore hold amongst the moments of the surface, and are of interest because $2\beta_2 - 3\beta_1 - 6 = 0$ is the condition that the margins should be Type III Curves; $q_{21} = r\sqrt{\beta_{10}}$, $q_{12} = r\sqrt{\beta_{01}}$ are the conditions that the regression should be linear; and $r = 1$ means absolute causation. We have pointed out above, that we shall only get approximately linear regression with this type of surface, when there is high correlation between the characters, and we saw that when this happens the margin curves will approach Type III Curves.

4. *To find* z_0.

The total frequency N is $\iint z\,dx\,dy$, where the integration is taken over the whole of the (xy) plane for which frequency exists, i.e. over that part contained between the bounding lines

$$1 - x/a + y/b = 0, \quad 1 + x/a' - y/b' = 0.$$

Let us integrate first with regard to x. We have

$$z_y = \int_{x_1}^{x_2} z_0 e^{-lx-my} (1 - x/a + y/b)^p (1 + x/a' - y/b')^{p'} dx,$$

the limits x_1, x_2 being given by

$$1 - \frac{x_2}{a} + \frac{y}{b} = 0, \quad 1 + \frac{x_1}{a'} - \frac{y}{b'} = 0,$$

y being regarded as constant.

Let us change the origin of x to x_1, i.e. replace x by $x + x_1$ in the above; then we have

$$z_y = \int_0^{x_2 - x_1} z_0 e^{-l(x+x_1)-my} \left(\frac{x}{a'}\right)^{p'} \left(1 - \frac{x + x_1}{a} + \frac{y}{b}\right)^p dx.$$

Now
$$x_1 = \left(\frac{y}{b'} - 1\right) a',$$

$$x_2 = \left(1 + \frac{y}{b}\right) a,$$

$$\therefore \quad x_2 - x_1 = a + a' + y\left(\frac{a}{b} - \frac{a'}{b'}\right) = aa'\left(\frac{1}{a} + \frac{1}{a'} + yX\right)$$

$$= aa' . u. \quad \text{(See p. 358.)}$$

And
$$1 - \frac{x_1}{a} + \frac{y}{b} = 1 + \frac{y}{b} - \frac{a'}{a}\left(\frac{y}{b'} - 1\right) = a'\left(\frac{1}{a} + \frac{1}{a'} + yX\right) = a'u.$$

Our integral is therefore

$$z_y = \int_0^{aa'u} z_0 e^{-lx-lx_1-my} \left(\frac{x}{a'}\right)^{p'} \left(a'u - \frac{x}{a}\right)^p dx$$

$$= z_0 e^{-lx_1-my} \int_0^{aa'u} e^{-lx} \left(\frac{x}{a'}\right)^{p'} (a'u)^p \left(1 - \frac{x}{aa'u}\right)^p dx.$$

Call
$$\frac{x}{aa'u} = t.$$

Then
$$z_y = z_0 e^{-lx_1-my} \int_0^1 e^{-laa'ut} (aut)^{p'} (a'u)^p (1-t)^p \, aa'u dt,$$

i.e.
$$z_y = z_0 e^{-lx_1-my} a^{p'+1} a'^{p+1} u^{p+p'+1} \int_0^1 e^{-laa'ut} t^{p'} (1-t)^p \, dt$$

$$= z_0 e^{-lx_1-my} a^{p'+1} a'^{p+1} u^{p+p'+1} \int_0^1 t^{p'} (1-t)^p \, dt \left(1 - laa'ut + \frac{(laa'ut)^2}{2!} - \ldots\right) dt.$$

We can obtain this integral as an infinite series, viz.

$$z_y = z_0 e^{-lx_1-my} a^{s'} a'^{s} u^{R-1} \left[\mathrm{B}(p+1, p'+1) - laa'u\mathrm{B}(p+1, p'+2) \right.$$
$$\left. + \frac{(laa'u)^2}{2!} \mathrm{B}(p+1, p'+3) - \ldots \right],$$

where the B-functions are the ordinary Beta functions.

Therefore

$$z_y = z_0 e^{-lx_1 - my} a^{s'} a'^{s} u^{R-1} \left[\frac{\Gamma(s)\Gamma(s')}{\Gamma(s+s')} - laa'u \frac{\Gamma(s)\Gamma(s'+1)}{\Gamma(s+s'+1)} + \frac{(laa'u)^2}{2!} \frac{\Gamma(s)\Gamma(s'+2)}{\Gamma(s+s'+2)} - \dots \right]$$

$$= z_0 e^{-lx_1 - my} a^{s'} a'^{s} u^{R-1} \frac{\Gamma(s)\Gamma(s')}{\Gamma(R)} \left[1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots \right],$$

which agrees with the result on p. 360, for

$$lx_1 + my = \frac{la'}{b'} y - la' + my$$

$$= y\left(\frac{p}{b} - \frac{p'}{b'} - \frac{pa'}{ba} + \frac{p'}{b'}\right) - la'$$

$$= ypa'X + \frac{pa'}{a} - p'a = pa'\left(u - \frac{1}{a'}\right) - p'$$

$$= pa'u - (R-2).$$

Therefore

$$z_y = z_0 a^{s} a'^{s} \frac{\Gamma(s)\Gamma(s')}{\Gamma(R)} e^{R-2} e^{-pa'u} u^{R-1} \left(1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots\right).$$

Then we wish to integrate z_y from the value of y given by $1 - x/a + y/b = 0$, $1 + x/a' - y/b' = 0$ to ∞. The value of y where these lines intersect is given by

$$1/a + 1/a' + yX = 0, \quad \text{i.e. } u = 0,$$

and we can therefore effect our summation by considering this integration with regard to u instead of y, viz.:

$$N = \int z_y dy,$$

the limits of integration being as stated above; this gives

$$N = \int_0^\infty z_y \frac{du}{X}$$

$$= \int_0^\infty z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} e^{R-2} e^{-pa'u} u^{R-1} \left(1 - laa'u \frac{s'}{R} + \frac{(laa'u)^2}{2!} \frac{s'.s'+1}{R.R+1} - \dots\right) du.$$

Let us write now $pa'u = v$.

$$N = \int_0^\infty z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2} e^{-v}}{(pa')^R} v^{R-1} \left\{1 - \frac{la}{p}\frac{s'}{R} v + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{R.R+1} \frac{v^2}{2!} - \dots\right\} dv.$$

We can write down this integral, it gives

$$N = z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2}}{(pa')^R} \left\{\Gamma(R) - \frac{las'}{pR}\Gamma(R+1) + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{R.R+1} \frac{\Gamma(R+2)}{2!} - \dots\right\}$$

$$= z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X\Gamma(R)} \frac{e^{R-2}}{(pa')^R} \Gamma(R) \left\{1 - \frac{las'}{p} + \left(\frac{la}{p}\right)^2 \frac{s'.s'+1}{2!} - \dots\right\}$$

$$= z_0 a^{s'} a'^{s} \frac{\Gamma(s)\Gamma(s')}{X} \frac{e^{R-2}}{(pa')^R} \left(1 + \frac{la}{p}\right)^{-s'},$$

and $$1+\frac{la}{p}=\frac{ap'}{a'p}.$$

$$\therefore\quad N=z_0a^sa'^{s'}\frac{\Gamma(s)\,\Gamma(s')\,e^{R-2}}{X\,(a'p)^s\,(ap')^{s'}}$$

$$=z_0e^{R-2}\frac{\Gamma(s)\,\Gamma(s')}{Xp^sp'^{s'}}.$$

$$\therefore\quad z_0=\frac{NXp^sp'^{s'}}{e^{R-2}\Gamma(s)\,\Gamma(s')}\qquad\text{.....(38),}$$

from which we can obtain z_0.

5. We finally require an approximation to the integral $I=\int_{x-h/2}^{x+h/2}\int_{y-k/2}^{y+k/2}z\,dx\,dy$ where $$z=z_0e^{-lx-my}(1-x/a+y/b)^p\,(1+x/a'-y/b')^{p'},$$ when h, k are subranges of x and y, small compared with σ_x, σ_y respectively.

We can express $\iint z\,dx\,dy$, where the integration is between the limits shewn above, in the form

$$\iint\left\{z+\xi\frac{dz}{dx}+\eta\frac{dz}{dy}+\frac{1}{2}\left(\xi^2\frac{d^2z}{dx^2}+2\xi\eta\,\frac{d^2z}{dx\,dy}+\eta^2\frac{d^2z}{dy^2}\right)+\ldots\right\}d\xi\,d\eta$$

where z, $\dfrac{dz}{dx}$, $\dfrac{dz}{dy}$, etc., in the above are understood to refer to the mid point (x, y) of the region of x and y enclosed in the rectangle $x-\dfrac{h}{2}$, $y-\dfrac{k}{2}$ to $x+\dfrac{h}{2}$, $y+\dfrac{k}{2}$, over which the summation is to take place, so that ξ, η vary from $-\dfrac{h}{2}$ to $\dfrac{h}{2}$; $-\dfrac{k}{2}$ to $\dfrac{k}{2}$ respectively.

We have $$\iint\xi^{2r}\eta^{2s}d\xi\,d\eta=\left[\frac{\xi^{2r+1}}{2r+1}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s+1}}{2s+1}\right]_{-k/2}^{k/2}$$

$$=4\left(\frac{h}{2}\right)^{2r+1}\left(\frac{k}{2}\right)^{2s+1}\frac{1}{(2r+1)(2s+1)}.$$

$$\iint\xi^{2r}\eta^{2s-1}d\xi\,d\eta=\left[\frac{\xi^{2r+1}}{2r+1}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s}}{2s}\right]_{-k/2}^{k/2}=0.$$

$$\iint\xi^{2r-1}\eta^{2s}d\xi\,d\eta=\left[\frac{\xi^{2r}}{2r}\right]_{-h/2}^{h/2}\left[\frac{\eta^{2s+1}}{2s+1}\right]_{-k/2}^{k/2}=0,\text{ etc.}$$

And we have for the integral I,

$$hk\left[z+\frac{1}{24}\left(h^2\frac{d^2z}{dx^2}+k^2\frac{d^2z}{dy^2}\right)+\frac{1}{192}\left(\frac{h^4}{10}\frac{d^4z}{dx^4}+\frac{h^2k^2}{3}\,\frac{d^4z}{dx^2dy^2}+\frac{k^4}{10}\frac{d^4z}{dy^4}\right)+\ldots\right].$$

Now $$\frac{1}{z}\frac{dz}{dx}=-l-\frac{p/a}{1-x/a+y/b}+\frac{p'/a'}{1+x/a'-y/b'}.$$

$$\therefore\quad\frac{1}{z}\frac{d^2z}{dx^2}-\left(\frac{1}{z}\frac{dz}{dx}\right)^2=-\frac{p/a^2}{(1-x/a+y/b)^2}-\frac{p'/a'^2}{(1+x/a'-y/b')^2}.$$

Let us call
$$1 - x/a + y/b = L,$$
$$1 + x/a' - y/b' = M.$$

Then
$$\frac{1}{z}\frac{d^2z}{dx^2} + \frac{1}{z}\frac{d^2z}{dy^2} = \left(-l - \frac{p/a}{L} + \frac{p'/a'}{M}\right)^2 + \left(-m + \frac{p/b}{L} - \frac{p'/b'}{M}\right)^2 - \frac{p/a^2}{L^2} - \frac{p'/a'^2}{M^2} - \frac{p/b^2}{L^2} - \frac{p'/b'^2}{M^2}$$
$$= l^2 + m^2 + 2l\left(\frac{p/a}{L} - \frac{p'/a'}{M}\right) - 2m\left(\frac{p/b}{L} - \frac{p'/b'}{M}\right) + \frac{p^2/a^2 + p^2/b^2 - p/a^2 - p/b^2}{L^2}$$
$$+ \frac{p'^2/a'^2 + p'^2/b'^2 - p'/a'^2 - p'/b'^2}{M^2} - \frac{2pp'}{LM}\left(\frac{1}{aa'} + \frac{1}{bb'}\right).$$

But
$$l^2 + m^2 = p^2/a^2 + p^2/b^2 + p'^2/a'^2 + p'^2/b'^2 - 2pp'\,(1/aa' + 1/bb')$$
$$- mp/b + lp/a = -p^2/a^2 + pp'/aa' - p^2/b^2 + pp'/bb'$$
$$- lp'/a' + mp'/b' = +pp'/aa' - p'^2/a'^2 + pp'/bb' - p'^2/b'^2.$$

$$\therefore \quad \frac{1}{z}\frac{d^2z}{dx^2} + \frac{1}{z}\frac{d^2z}{dy^2}$$
$$= \left(\frac{p^2}{a^2} + \frac{p^2}{b^2}\right)\left\{\frac{1}{L^2} - \frac{2}{L} + 1\right\} + \left(\frac{p'^2}{a'^2} + \frac{p'^2}{b'^2}\right)\left\{\frac{1}{M^2} - \frac{2}{M} + 1\right\}$$
$$- 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left\{\frac{1}{LM} - \frac{1}{L} - \frac{1}{M} + 1\right\} - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{p}{L^2} - \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\frac{p'}{M^2}$$
$$= \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left\{p^2\left(1 - \frac{1}{L}\right)^2 - \frac{p}{L^2}\right\} + \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\left\{p'^2\left(1 - \frac{1}{M}\right)^2 - \frac{p'}{M^2}\right\}$$
$$- 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left(1 - \frac{1}{L}\right)\left(1 - \frac{1}{M}\right).$$

Thus the frequency on a square, when $h = k = 1$, is
$$z\left[1 + \frac{1}{24}\left\{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(p^2\left(1 - \frac{1}{L}\right)^2 - \frac{p}{L^2}\right) + \left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)\left(p'^2\left(1 - \frac{1}{M}\right)^2 - \frac{p'}{M^2}\right)\right.\right.$$
$$\left.\left. - 2pp'\left(\frac{1}{aa'} + \frac{1}{bb'}\right)\left(1 - \frac{1}{L}\right)\left(1 - \frac{1}{M}\right)\right\}\right],$$
to the first approximation, where z is the middle ordinate, and L, M are the values of $1 - x/a + y/b$, $1 + x/a' - y/b'$ for x, y, the centres of the square, sides $h = k = 1$.

We have here a formula for obtaining a frequency from the middle ordinate in terms of the mid-ordinate with a correction, which will be small, so long as the base (hk) for which we are obtaining the frequency is small compared with the standard deviations of the variables.

6. *Practical work.*

The data I had before me with which to illustrate the results of the preceding theory were the data referred to before, contained in the results of 2922 daily observations of the barometric heights at Southampton and Laudale. The moments of this correlation table had been worked out previously by Professor Pearson in connection with some other work, and I have to thank him for his kindness in permitting me to use these results, which have saved me much labour.

Referring to the Southampton variate as x and the Laudale as y the results are as follows:

$$\sigma_x = 3{\cdot}250067, \quad \sigma_y = 3{\cdot}932290, \quad r = {\cdot}780225,$$
$$\beta_{10} = {\cdot}171140, \quad \beta_{01} = {\cdot}224536,$$
$$q_{21} = {\cdot}286962, \quad q_{12} = {\cdot}310386.$$

The unit of working is 0″·1.

The preliminary work is the finding of θ, ϕ, λ, s from equations (35), (36), (37), and as the elimination of all but one of these constants from the equations would mean a stupendous piece of algebra, it was thought better to find them by a process of approximation. As a help to locating the values of θ and ϕ we observe that $l = -\frac{p}{a} + \frac{p'}{a'} = \frac{p}{a}(\theta - 1)$ and $m = \frac{p}{b} - \frac{p'}{b'} = \frac{p}{b}(1 - \phi)$; as for large values of x and y the frequency is zero, we shall have l and m positive, and therefore we shall have $\theta > 1 > \phi$. Further, in order to get the four constants λ, θ, ϕ, s we really only need four of the momental constants, r, β_{10}, β_{01}, q_{21}, q_{12}; but since if we use q_{21} and not q_{12} we may be giving greater weight to one part of the table than to another, it is better to form one equation from the two equations (37), and obtain in this way four equations for our constants. We obtained these results:

$$\lambda = 1{\cdot}155645, \quad s = 11{\cdot}88185, \quad \theta = 2{\cdot}831825, \quad \phi = {\cdot}680439.$$

From these we had

$$p = 10{\cdot}88185\ , \quad p' = 12{\cdot}73120,$$
$$a = 17{\cdot}34984\ , \quad a' = 7{\cdot}167960,$$
$$b = 8{\cdot}817063, \quad b' = 15{\cdot}160064,$$
$$l = 1{\cdot}148924, \quad m = {\cdot}394396,$$

and finally from (38) $z_0 = 64{\cdot}87113$,

and from [21 (i)] $\bar{x} = 1{\cdot}24537, \quad \bar{y} = 1{\cdot}44314.$

The mean for the Southampton (x) distribution being at 29″·9839, and that for the Laudale (y) distribution being at 29″·8488, the mode and origin of our surface is at 30″·10845 (Southampton), 29″·99315 (Laudale).

We are now in a position to calculate the mid-ordinates of our theoretical surface corresponding to the observed frequencies, and by using the approximate formula developed above, we can obtain the theoretical frequencies with reasonable accuracy. These are shewn in Table I with the observed frequencies. The fact that the total frequency of the theoretical surface agrees within 0·1 % of the observed frequency is to a certain extent a measure of the reasonableness of this approximation to the individual theoretical frequencies. A comparison of the observed and theoretical frequencies in Table I shews that the surface fits reasonably well the observed data. To shew this fit further the theoretical surface has been constructed in model form from the mid-ordinates. I owe this model to Miss H. G. Jones, whom I would like to thank here for her kindness and patience

TABLE I. *Frequency of Barometric Heights at Southampton and Laudale (2922 days observations). Actual and Theoretical Frequencies.*

Height of Barometer at Southampton in Inches.

Laudale in Inches.	31·0	30·9	30·8	30·7	30·6	30·5	30·4	30·3	30·2	30·1	30·0	29·9	29·8	29·7	29·6	29·5	29·4	29·3	29·2	29·1	29·0	28·9	28·8	28·7	28·6	28·5	Totals
31·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	*0·1*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*
30·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	0·1	*0·3*	*0·4*	*0·3*	*0·1*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*1·2*
30·8	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
	—	*0·4*	*1·2*	*1·6*	*1·1*	*0·4*	*0·1*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*4·8*
30·7	—	1	—	4·5	4	3	1·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
	—	*0·3*	*1·7*	*3·8*	*4·5*	*3·1*	*1·3*	*0·4*	*0·1*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*15·2*
30·6	—	1	3	7·75	9·75	8·5	3·5	2·25	0·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
	—	*0·1*	*0·7*	*5·2*	*9·5*	*9·9*	*6·6*	*2·9*	*1·0*	*0·2*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*36·8*
30·5	—	1	0·5	6·25	9·25	17	15·75	6·5	5·5	1·75	0·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	64
	—	—	*0·2*	*4·6*	*12·6*	*18·9*	*17·7*	*11·4*	*3·3*	*1·8*	*0·5*	*0·1*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*71·6*
30·4	—	—	0·5	5·75	11·75	22	24·75	39·75	19·75	12·25	4·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	141
	—	—	—	*2·6*	*11·3*	*24·1*	*30·8*	*26·6*	*16·7*	*7·9*	*2·8*	*0·8*	*0·2*	—	—	—	—	—	—	—	—	—	—	—	—	—	*124*
30·3	—	—	—	5·25	11·5	15·5	24·75	55	50	25·5	6	5	1·25	0·25	—	—	—	—	—	—	—	—	—	—	—	—	200
	—	—	—	*1·0*	*7·1*	*22·0*	*38·1*	*42·8*	*34·4*	*20·9*	*9·9*	*3·7*	*1·1*	*0·3*	—	—	—	—	—	—	—	—	—	—	—	—	*181·3*
30·2	—	—	—	1	4·25	21·75	24	38·5	55·5	66	32	12·5	4·75	2·75	—	—	—	—	—	—	—	—	—	—	—	—	263
	—	—	—	*0·2*	*3·2*	*15·0*	*35·3*	*51·3*	*51·8*	*39·2*	*23·2*	*11·0*	*4·3*	*1·4*	*0·4*	*0·1*	—	—	—	—	—	—	—	—	—	—	*236·4*
30·1	—	—	—	—	2	10·75	21·75	23·75	55·75	55·5	46·75	26	15	3·25	—	—	—	—	—	—	—	—	—	—	—	—	260·5
	—	—	—	—	*1·0*	*7·7*	*25·6*	*48·1*	*60·4*	*55·8*	*39·9*	*23·0*	*10·9*	*4·3*	*1·4*	*0·4*	*0·1*	—	—	—	—	—	—	—	—	*278·6*
30·0	—	—	—	—	—	5	11	26·25	48·25	63	58	27·5	24·5	10	1·5	1·5	1	—	—	—	—	—	—	—	—	—	277·5
	—	—	—	—	*0·3*	*3·0*	*14·0*	*36·0*	*56·8*	*63·4*	*54·1*	*36·9*	*20·6*	*9·8*	*4·0*	*1·4*	*0·4*	*0·1*	—	—	—	—	—	—	—	—	*300·8*
29·9	—	—	—	—	—	4	4	19	31·75	54·25	63·25	57	27·5	11·5	8·75	1	0·5	1	—	—	—	—	—	—	—	—	283·5
	—	—	—	—	—	*0·8*	*6·4*	*21·6*	*43·9*	*59·5*	*60·2*	*48·0*	*31·3*	*17·2*	*8·1*	*3·3*	*1·2*	*0·4*	*0·1*	—	—	—	—	—	—	—	*302*
29·8	—	—	—	—	—	—	7	13	24·25	41·75	59·75	48	36·75	18	17·5	6·5	3·5	0·5	—	1	—	—	—	—	—	—	277·5
	—	—	—	—	—	*0·2*	*2·2*	*10·8*	*28·2*	*47·0*	*56·3*	*52·2*	*39·2*	*24·6*	*13·2*	*6·2*	*2·6*	*0·9*	*0·3*	*0·1*	—	—	—	—	—	—	*284*
29·7	—	—	—	—	—	—	2	8	12	36·5	41·25	47	42·25	19·5	16	7·5	5	5	2·5	0·5		—	—	—	—	—	245
	—	—	—	—	—	—	*0·6*	*4·3*	*15·1*	*31·6*	*44·8*	*48·7*	*41·8*	*29·8*	*18·1*	*9·6*	*4·4*	*1·8*	*0·7*	*0·2*	*0·1*	—	—	—	—	—	*251·6*
29·6	—	—	—	—	—	—	0·5	4	8·5	16	26·5	47	32·75	25·5	23·75	12·5	8	3	1	3	—	—	—	—	—	—	212
	—	—	—	—	—	—	*0·1*	*1·4*	*6·8*	*18·2*	*31·4*	*39·6*	*38·9*	*31·2*	*21·2*	*12·6*	*6·5*	*3·0*	*1·2*	*0·5*	*0·2*	—	—	—	—	—	*212·8*
29·5	—	—	—	—	—	—	—	1	3	13	18	31	45	39·5	20	10·5	8	2·5	—	0·5	—	—	—	—	—	—	192
	—	—	—	—	—	—	—	*0·3*	*2·5*	*8·9*	*19·1*	*28·2*	*31·8*	*28·9*	*22·1*	*14·3*	*8·2*	*4·2*	*1·9*	*0·8*	*0·3*	*0·1*	—	—	—	—	*171·6*
29·4	—	—	—	—	—	—	—	—	0·5	5	13·5	17·25	24·5	23·75	18	13·5	7·25	5·25	3	0·5	1	1	1	—	—	—	135
	—	—	—	—	—	—	—	—	*0·8*	*3·7*	*10·0*	*17·3*	*23·2*	*23·8*	*20·1*	*14·5*	*9·1*	*5·1*	*2·5*	*1·1*	*0·5*	*0·2*	*0·1*	—	—	—	*132*

Height of Barometer at	1	4	4	30·5	52·5	107·5	140·5	237	315	395·5	382·5	339·5	288	201	150·5	98·5	65	50	23·5	15·5	7·5	7	3	—	1	2	
	0·1	*1·2*	*5·6*	*19·3*	*50·7*	*105·1*	*178·8*	*257·9*	*322*	*359·9*	*359·4*	*327·5*	*275·3*	*215·1*	*157·7*	*108·9*	*71·4*	*44·7*	*26·4*	*15·0*	*8·5*	*4·5*	*2·3*	*1·0*	*0·5*	—	*2918·8*
29·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	97·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*98·8*
29·2	—	—	—	—	—	—	—	—	*0·2*	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	67·5
	—	—	—	—	—	—	—	—	—	—	—	—	*0·4*	*14*	*13*	*8*	*7*	*3*	*2*	—	—	—	—	—	—	—	*71·3*
29·1	—	—	—	—	—	—	—	—	—	—	—	—	1	*11·8*	*12·5*	*11·0*	*8·3*	*5·5*	*3·2*	*1·6*	*0·8*	*0·4*	*0·1*	—	—	—	63
	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	10·5	10	12·5	6	9·5	3	—	1	1	—	—	—	—	*50*
29·0	—	—	—	—	—	—	—	—	—	1	—	4	4	3	5·75	3·75	6·5	6	4·5	2·5	1·5	—	—	—	—	—	38·5
	—	—	—	—	—	—	—	—	—	*0·2*	*0·8*	*2·2*	*0·9*	*3·9*	*5·4*	*5·8*	*5·3*	*4·1*	*2·8*	*1·7*	*1·0*	*0·5*	*0·2*	*0·1*	—	—	*34*
28·9	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2·75	5·75	2·5	4·5	4	—	2	1	—	—	—	—	24·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	*2·0*	*3·1*	*3·7*	*3·7*	*3·2*	*2·3*	*1·5*	*0·9*	*0·5*	*0·3*	*0·1*	*0·1*	—	*22·6*
28·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2·5	2	1·5	0·5	0·5	—	3	—	—	—	—	11
	—	—	—	—	—	—	—	—	—	—	—	—	*0·3*	*0·9*	*1·6*	*2·2*	*2·5*	*2·3*	*1·8*	*1·3*	*0·8*	*0·5*	*0·3*	*0·1*	*0·1*	—	*14·7*
28·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	1·5	—	1·5	1	3	—	—	—	—	—	—	7·5
	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·8*	*1·2*	*1·5*	*1·5*	*1·3*	*1·0*	*0·7*	*0·4*	*0·2*	*0·1*	*0·1*	—	*9·2*
28·6	—	—	—	—	—	—	—	—	—	—	—	—	1	—	0·5	1	—	1	—	1	—	—	—	—	—	—	4·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·6*	*0·9*	*1·0*	*0·9*	*0·7*	*0·5*	*0·3*	*0·2*	*0·1*	*0·1*	—	*5·7*
28·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·3*	*0·4*	*0·6*	*0·6*	*0·5*	*0·4*	*0·3*	*0·2*	*0·1*	*0·1*	—	*3·6*
28·4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	1	—	—	1	2·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·2*	*0·3*	*0·3*	*0·3*	*0·3*	*0·2*	*0·1*	*0·1*	—	—	*1·9*
28·3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	0·5
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·3*	*0·2*	*0·2*	*0·2*	*0·1*	*0·1*	*0·1*	—	—	*1·2*
28·2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·1*	*0·1*	*0·1*	*0·1*	*0·1*	—	—	—	*0·6*
28·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	*0·1*	*0·1*	*0·1*	*0·1*	—	—	—	—	*0·4*
28·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
27·9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Totals	1	4	4	30·5	52·5	107·5	140·5	237	315	395·5	382·5	339·5	288	201	150·5	98·5	65	50	23·5	15·5	7·5	7	3	—	1	2	2922

Theoretical frequencies in italics.

and good-will in undertaking such a hard task. As the observed data had been shewn in model form many years ago by Professor Pearson*, we are enabled to compare the two, as we can compare data involving only one variable. The accompanying plate shews the two models and a "composite" photograph of them superposed. Finally we can obtain a further comparison of the theoretical with the observed by means of the Goodness of Fit Test. Table II shews the marginal arrays compared, with fairly good results.

When the Laudale and Southampton marginal totals are fitted by the usual four moment method, the theoretical curves when compared with the observations by means of the Goodness of Fit Test give P's of ·62 and ·59 respectively. I am indebted for these figures to Mr E. S. Pearson.

TABLE II.

Comparison of Actual and Theoretical Frequencies of Barometric Heights at Southampton and Laudale, from the Marginal Totals of Table I.

Laudale.

Barometric Height	Observed	Theoretical
Above 30·6	16	21·3
30·6	36	36·8
30·5	64	71·6
30·4	141	124·0
30·3	200	181·3
30·2	263	236·4
30·1	260·5	278·6
30·0	277·5	300·8
29·9	283·5	302·0
29·8	277·5	284·0
29·7	245	251·6
29·6	212	212·8
29·5	192	171·6
29·4	135	132·0
29·3	97·5	98·8
29·2	67·5	71·3
29·1	63	50·0
29·0	38·5	34·0
28·9	24·5	22·6
28·8	11	14·7
Below 28·8	17	22·6
Totals	2922	2918·8

$P = ·4$.

Southampton.

Barometric Height	Observed	Theoretical
Above 30·7	9	6·9
30·7	30·5	19·3
30·6	52·5	50·7
30·5	107·5	105·1
30·4	140·5	178·8
30·3	237	257·9
30·2	315	322·0
30·1	395·5	359·9
30·0	382·5	359·4
29·9	339·5	327·5
29·8	288	275·3
29·7	201	215·1
29·6	150·5	157·7
29·5	98·5	108·9
29·4	65	71·4
29·3	50	44·7
29·2	23·5	26·4
29·1	15·5	15·0
Below 29·1	20·5	16·8
Totals	2922	2918·8

$P = ·2$.

To get the comparison between the theoretical and observed data for the whole of the surface we had to group some of the cell-contents together, for if we had used the χ^2 test in the material as shewn in Table I we should have about 170 groups. In this connection we may recall an observation by Professor Pearson in

* [1897. It is a "pseudo-frequency surface," in which frequencies have been used as ordinates, instead of shewing them by the more recent and reasonable method of briquettes. Ed.]

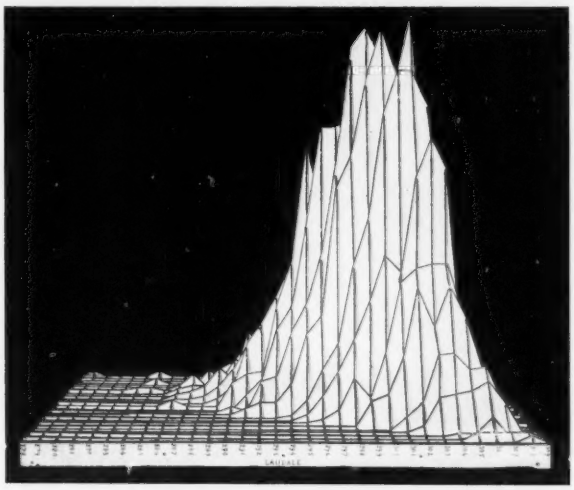


Fig. 1.

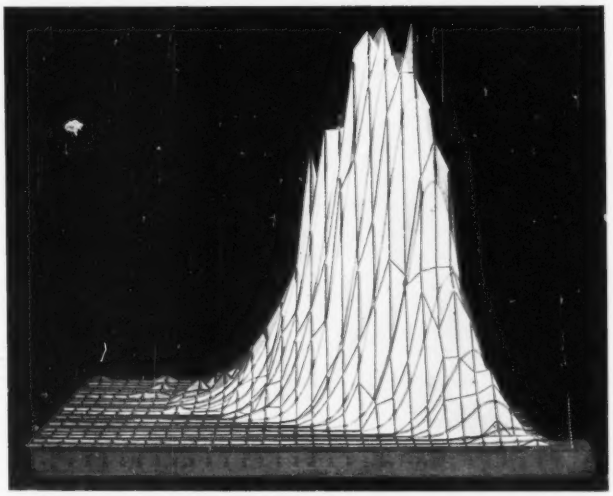


Fig. 2.

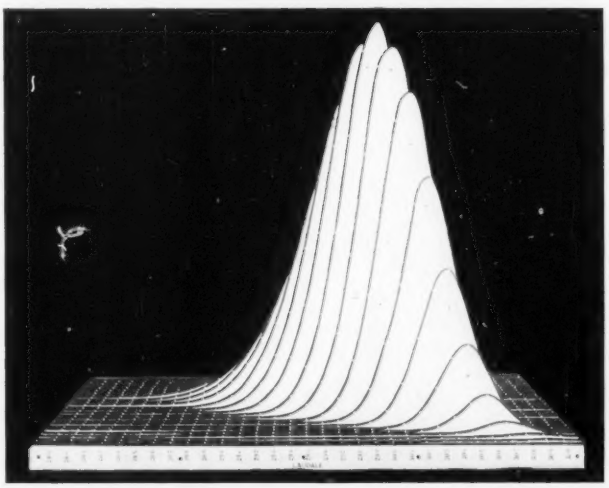


Fig. 3.

Figures 1 and 3 shew the actual and theoretical surfaces, and Figure 2 shews the two superimposed by means of a composite photograph.

his original paper on the Goodness of Fit Test (*Phil. Mag.* Vol. L. (1900), p. 157*) to the effect that one should take a moderate number of groups in using the Test, otherwise when the grouping is too fine the value of χ^2 becomes imponderable and cannot be used to give any result, while when the grouping is too broad, until finally there is only one group, the value of χ^2 becomes zero and the fit is perfect. I have endeavoured to perform the necessary grouping in as systematic manner as possible as shewn by the thick lines in Table I. The results of the grouping are shewn in Table III where there are 24 groups. The resulting P is ·04, which is

TABLE III.

(Obtained from Table I.) Shewing Actual and Theoretical Frequencies with Wider Grouping.

Southampton.

Laudale.

	30·7—	30·6—30·4	30·3—30·1	30·0—29·8	29·7—29·5	29·4—29·2	29·1—
30·6—	19·25 *16·9*	30·25 *36·6*					
30·5—30·3		171·5 *191·7*	218·5 *170·4*	17·25 *19·1*			
30·2—30·0		118·5 *115·5*	432·5 *462·8*	247·0 *223·9*	20·25 *24·1*		
29·9—29·7			240·5 *262·0*	422·75 *422·5*	106·25 *130·1*		
29·6—29·4			51·5 *42·7*	255·5 *239·5*	187·0 *188·7*	64·5 *58·3*	
29·3—29·1				72·5 *59·2*	108·5 *107·0*	45·0 *49·5*	8·0 *9·2*
29·0—28·8					29·0 *32·4*	36·5 *38·3*	11·5 *10·0*
28·7—							8·0 *8·4*

(24 groups.) $P = ·04$. Theoretical figures in italics.

not as good a result as had been looked for, but which shews that the theoretical surface, even from the point of view of the χ^2 test, is a fair fit to the observed table of frequency.

7. An effort has been made here to find a skew surface which would fit reasonably well certain data, and to a certain extent the form of the surface was guessed at from the given distribution. For purposes of calculation it was needful that

* [The reference appears to be to p. 160, but surely the warning there refers to grouping in such fine intervals that the distribution of the cell-content follows a Poisson's series and not a normal curve, i.e. to cells with a few units in them? Otherwise the number of groups need not be small. ED.]

the required theoretical surface should have as simple an equation as possible. The equation to the surface we have considered is reasonably simple, and although it might be asserted that it is empirical, it appears to fit fairly well the observed data. The form assumed for the surface is not however entirely empirical, for it can be obtained as a special case from the general differential equations obtained from a fundamental theorem in probability, concerned with the problem of the drawing of two samples from a certain population, where correlation arises as a result of making these two samples have a part which is common to both. This method of trying to explain correlation in terms of chance problems, by expanding Bayes' Theorem into two dimensions, is itself purely arbitrary, but it seems the logical step and leads to differential equations which appear to be the type of differential equations we require for the two-dimensional surface. Having obtained the type of differential equation by this method, it seems preferable, at any rate in the beginning of such work, to experiment with different kinds of differential equations which can be immediately obtained from the general differential equations, but which do not necessarily correspond directly to a particular chance problem.

This appears to be the most useful course to pursue, as a general solution to our equations seems impossible at first sight, and as a treatment of more simple equations leads to results which are of interest and involves methods which may be used later in the more general treatment. For instance at first I had in view, as the method of obtaining the moments of the surface, the consideration of such an integral as $\iint \frac{d^2z}{dxdy} x^m y^n \, dxdy$, but in the work in connection with this paper I saw that I could get the results more easily by considering such integrals as $\iint \frac{dz}{dx} x^m y^n \, dxdy$ and $\iint \frac{dz}{dy} x^m y^n \, dxdy$; further I saw the possibilities which lay before me in treating such integrals as $\int \frac{dz}{dx} x dx$, $\int \frac{dz}{dy} y dx$ etc., when considering the array totals and the means of the arrays. From these points of view I think that the foregoing treatment of the skew surface justifies itself, even though we cannot find any physical counterpart to the two straight lines which I have taken as the bounds to the surface.

I have discussed elsewhere the problem of finding a skew surface which has Pearson Curves as its arrays, and have shewn that such surfaces, except for particular forms, are symmetrical. The papers dealing with this problem have not yet been published.

In the course of the work on this subject of skew-correlation surfaces, I have come to the following conclusions:

(1) That we must consider the problem of fitting to observational data surfaces which are bimodal, or multimodal, in form, but unimodal in fact. It will be seen that the array curves of the surface discussed in this paper are bimodal.

(2) That the type of regression line which appears in skew correlation is only linear in very particular cases, and generally is of the form given by the equation

$$\frac{\bar{x}_y - \bar{x}}{\sigma_x} = \text{infinite series in } \left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right),$$

which may approximate, with given data, to a straight line, a parabola, a cubic etc.; but which, with simpler equations, may actually be a hyperbola of the form

$$\left(\frac{\bar{x}_y - \bar{x}}{\sigma_x} - \text{constant}\right)\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y} - \text{constant}\right) = \text{constant},$$

or a cubic of the form

$$\left(\frac{\bar{x}_y - \bar{x}}{\sigma_x}\right)\left[\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right)^2 + \text{const.}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right) + \text{const.}\right] = \text{const.} + \text{const.}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right).$$

This "cubical regression" in the usual sense of the term is merely a short way of saying "the best fitting cubic to the regression line," which is really of the form $\frac{\bar{x}_y - \bar{x}}{\sigma_x} = \text{infinite series in } \left(\frac{\bar{y}_x - \bar{y}}{\sigma_y}\right)$; but we may have real cubical regression, the cubic being of the form $x(ay^2 + by + c) = A + By$, or some such equation.

(3) The only homoscedastic system with linear regression is the Gaussian surface. I do not know if this has been shewn to be the case, but I feel sure that this is so.

A SUMMARY OF THE PRESENT POSITION WITH REGARD TO THE INHERITANCE OF INTELLIGENCE*.

By ETHEL M. ELDERTON, Galton Fellow, University of London.

INTELLIGENCE is defined in the Oxford Dictionary as meaning primarily "the faculty of understanding" and it is in that sense that I propose to use the word. Secondary meanings for the word intelligence have grown up, meanings that have rather the sense of extent of knowledge, and it is that idea that we must banish from our minds. People with good intelligence will be capable of acquiring information readily and will assimilate it; but it is possible to be a regular repository of information and yet to be singularly lacking in the power to use it; such power lies in intelligence, that faculty of using one's existing information to arrive at a right decision in some new field of thought or at a reasoned course of action under unusual circumstances, and it is this power which will make the difference between a well informed intelligent person and a well informed unintelligent one. We shall expect a correlation between intelligence and acquired knowledge but not by any means necessarily a very high one. Is this faculty of understanding inherited or is it largely, if not wholly, the product of opportunity, of environment and of education? No one would suggest that all the children in one school are equally intelligent but are the variations we see due to differences in natural ability or to a different home environment, or to an unsuitable educational environment, or are the variations due chiefly to one of such causes, while the others are of some but of much less importance?

Of recent years education has received much attention, new systems have been tried, conference has followed conference and the child normal and abnormal is constantly discussed, and one of the questions before us is whether the ideal system of education can create or even increase intelligence, or whether it can only give to child or man the material for his intelligence to play upon. The advocates of any new system of education or of any thing else new (even of Eugenics) are sometimes inclined to attribute possibilities to their new theories which are unlikely to be realized. No new scheme can be started without a great deal of enthusiasm on the part of the starters, enthusiasts must be whole hearted supporters of their theories and naturally claims will sometimes be advanced beyond their legitimate boundaries. If we find that this faculty of understanding is innate it does not for a moment follow that education is useless or that the form it takes matters not at all. The sharper the tool is ground the greater the advantage to those who can make use of it.

* Being a lecture delivered at the Galton Laboratory of National Eugenics.

The study of the inheritance of intelligence appealed very strongly to Francis Galton; it was the cheerful side of Eugenics, the inheritance of marked ability and of the best characteristics of the race rather than of the worse aspects of humanity, that chiefly attracted his attention. Galton has stated that it was the publication of the *Origin of Species* in 1859 that turned his thoughts to the question of heredity in man. Current views of inheritance at that time were, he said, very vague and contradictory; generally speaking most authors agreed that all bodily and some mental qualities were inherited by brutes but refused to believe the same of man. Even the word heredity in those days was considered fanciful and unusual and Galton was chaffed by a friend for adopting it from the French.

If we imagine individuals arranged in order of their intelligence as they can be in order of their height or any other characteristic that can be measured numerically, then we may say that it was the inheritance of giant intellect, of the intellect of the foremost man in 4000 as Galton says in *Hereditary Genius*, that chiefly appealed to him. He was not content that a man should be a Fellow of the Royal Society only, but he must be something more even than that, for he must have some additional title to fame before Galton put him in the first rank of English Men of Science. To show how much the cheerful side of Eugenics appealed to Francis Galton I need only remind the reader of his chief works, *Hereditary Genius*, 1869, *English Men of Science*, 1874, and to a lesser extent *Human Faculty*, 1883, and *Natural Inheritance*, 1889. The first work he suggested to the first Galton Fellow was a book on Noteworthy Families followed almost at once by a study of the Inheritance of Ability from the Oxford Class Lists. It may be urged that Galton proved that the highest form of intelligence was inherited, but I feel sure that he thought that further investigations of a more definite and numerical nature were required, that the study must be extended to every grade of intelligence and that even if we showed to his and our satisfaction that inheritance of intelligence was of vital importance, we still had to investigate whether such inheritance could be largely modified by environment, either physical or educational.

Before the publication of the first papers of the first Galton Fellow another attempt had been made to measure the inheritance of intelligence in what we can regard as ordinary individuals. This was organized by Professor Pearson and the data were collected from pairs of brothers and sisters in schools; the results were first given in the 4th Huxley Lecture and were re-published in 1904 in *Biometrika*, Vol. III. p. 131 et seq. The material was obtained from very many schools and provided information, not only as to the intelligence of the children but also as to their health; further various physical measurements were included.

I will explain the terms then used with regard to intelligence in some detail as the scale with one modification is the one we still use in the Galton Laboratory.

A. Very Dull. Capable of holding in their minds only the simplest facts and incapable of perceiving or reasoning about the relationship between facts.

B. Slow Dull. Capable of perceiving relationship between facts in some few fields with long and continuous effort, but not generally without much effort or without much external assistance.

C. Slow. Very slow in thought generally, but with time understanding is reached.

D. Slow Intelligent. Slow generally although possibly more rapid in certain fields.

E. Intelligent. Ready to grasp and capable of perceiving facts in most fields. Capable of understanding without much effort.

F. Quick Intelligent. Very bright and quick both in perception and acquirement and this not only of customary but of novel facts. Ready to reason rightly on purely self initiative. It will be noticed that in all these definitions it is the idea of understanding and of reason that is uppermost and not the idea of information acquired.

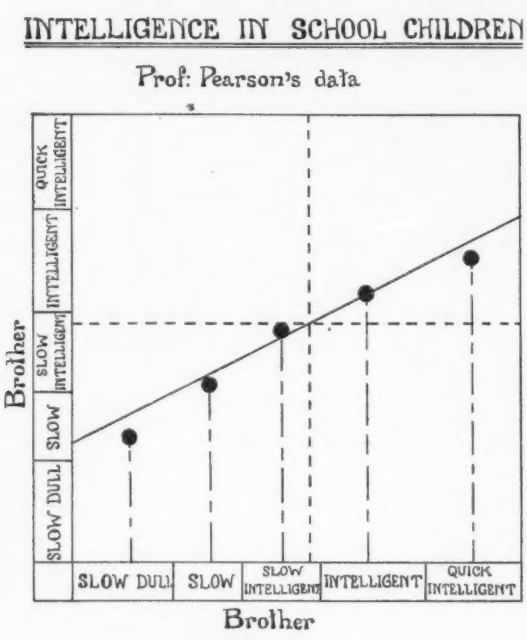


Diagram I.

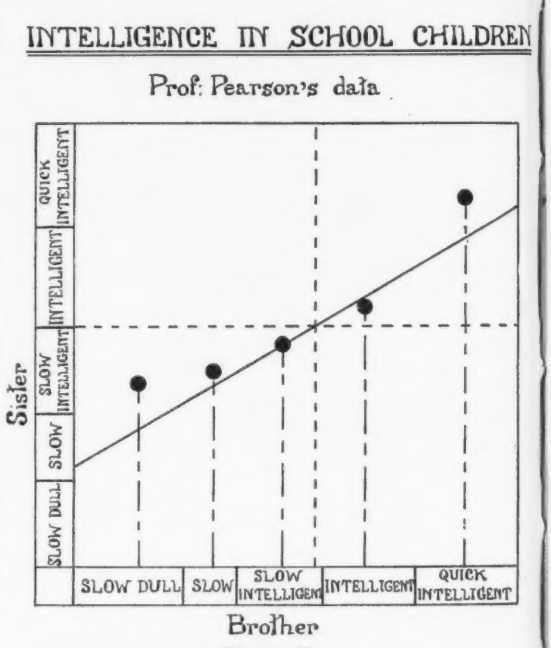


Diagram II.

Diagrams I and II show for brothers and for brothers and sisters how steadily the intelligence of the second member of a pair of "siblings" increases with the intelligence of the first. If one considers a group of boys with slow dull intelligence it will be seen that the average intelligence of the brothers of these boys is in the slow group and that the brothers do not even reach the middle of that group, but that when boys have been classed as intelligent their brothers also are found just in the intelligent group. Pairs of brothers and sisters show the same thing; the slow intelligent boys have sisters whose average intelligence is also slow intelligent

and quick intelligent boys have sisters who, on the average, have a high grade of intelligence.

We must now consider the coefficients of correlation in siblings for the physical as well as the mental characters which were obtained from Pearson's data; some of them are given in the following table:

TABLE I.

	Brothers	Sisters	Brother—Sister
Head Length ...	·50	·43	·46
Head Height ...	·55	·52	·49
Eye Colour ...	·54	·52	·53
Intelligence ...	·52	·50	·49

It will be seen from this table that all the coefficients of correlation are very similar in value; as we should lay no stress on differences in percentages of five or six so we lay no stress on differences in correlation coefficients of ·05 and ·06—they are insignificant. In considering these sibships it has been argued that we were dealing with widely different environments and that the brothers were alike because of the similarity of environment. The agreement between the results for intelligence and for physical characters seems to me to render the argument invalid but we will now see from the Oxford Class Lists, and from Charterhouse school where the environment is more uniform, how far intelligence runs in families*. We are assuming that intelligence and success in the final schools at Oxford are highly correlated and that intelligence and the form reached in a public school are also correlated; this I think is on the whole true, though naturally there will be many exceptions, but for the moment let us consider that we are discussing not intelligence but success. There are certain other characteristics that are required for success as well as intelligence, such as health and perseverance, but I doubt whether success in Oxford Finals could be achieved without intelligence. Here we will consider only fathers and sons for by so doing we obtain I think a very uniform environment. The sons took their degree before 1892; they all had

TABLE II.

Degree at Oxford taken by the Father	Percentage of Sons who took I and II Class Honours
I and II	27
III	15
Pass	12
None	9

* "The Inheritance of Ability," by E. Schuster and Ethel M. Elderton, *Eugenics Laboratory Memoirs*, I.

fathers at Oxford and though the environment would doubtless vary the variations would be comparatively slight. There would be no lack of the necessities of life and all the sons would come from cultured homes.

It will be seen that the number of sons who obtained first and second class honours falls steadily according to the degree taken by the father.

It is possible that two groups of men are included in this table. The first consisted of men who were placed in the position they deserved, men who tried for honours whether they obtained them or not and those who did not enter for honours because it was recognized that they were not up to the required standard, and the second group of men who did not try for honours though intellectually they may have been up to the standard. Family tradition and outside interests influence a man in deciding whether he will enter for honours or not and this applied even more thirty years ago than it does now.

At Charterhouse the boys are living under similar environments and though home conditions may vary, such variations will only be slight. The following table gives the percentage of brothers who were in groups I and II at ages 15, 16, and 17 according to the place in the school attained by a brother at that age. Group I includes the sixths, group II the fifths and so on down to group VII which includes

TABLE III.

Percentage of Brothers in Groups I and II.

	15 years	16 years	17 years
I	32·1	79·3	84·6
II		49·7	67·8
III	21·2	35·1	45·9
IV	15·0	21·9	42·8
V	9·7	19·0	28·6
VI	6·0	12·0	
VII	4·3		
Average	15·0	34·0	60·8

all boys in the shell. Of all boys of 15 years 15 per cent. are found in groups I and II but of the boys who were in groups I and II, 32 per cent. of their brothers reached that position at 15 years while of the boys who were in shell at 15 years only 4 per cent. of their brothers reached this highest grade. Pairs of brothers at ages 16 and 17 show the same thing, 34 per cent. of boys of 16 are in groups I and II and 61 per cent. of boys of 17 but the percentage of boys in the two highest groups varies from 79 to 12 at age 16 and from 85 to 29 at age 17 according to the position reached by the brother. The mean correlation coefficient for the three ages is ·48 which agrees very well with that found for brothers in the school data where the teachers estimated the intelligence.

It must be remembered that up to this time no-one had talked about tests for intelligence or about intelligence quotients. Binet did not publish his work on the subject till 1908, the year after the paper on the Oxford Class Lists was published, and the only way to judge intelligence was by estimating the character, using such a scale as I have described on p. 379. These estimates had to be made by those who might be supposed to know the children well, generally by the masters and mistresses. In order to test how far such an estimate could be relied upon Professor Pearson when dealing with intelligence in school children had asked two or three different teachers in several schools to apply the classification to between 30 and 50 pupils known to each of them. Classifications were to be made absolutely independently, often by teachers of different subjects. Just under 85 per cent. of the children were put into the same classes by the different teachers while another 10 per cent. differed by only one class. Further investigations as to the adequacy of the teacher's estimate were made later. Teachers in four different Aberdeen schools judged the mental capacity of 249 boys in four groups, excellent, good, moderate, and dull. Each boy's examination place was given and Miss H. G. Jones in 1909* compared the teacher's estimate of intelligence with the place attained in the examination. It may be argued that an examination is not a perfect test of intelligence and I quite agree; an examination is supposed to test information as well as intelligence but unless it is very badly conducted the more intelligent people will be found near the top and the least intelligent at the bottom of the list. The place in the examination was divided by the number of boys in the class but perhaps the easiest way in which we can see the result is by giving the average position that would be attained out of 100 boys by those of excellent, good, moderate, and dull intelligence. The average position attained by boys of excellent intelligence in 100 boys is fifteenth, by boys of good intelligence thirty-fourth, by boys of moderate intelligence sixty-second and by boys of bad intelligence eighty-eighth, which shows that the teacher's estimate and the result of an examination test agree very well.

In 1911 Dr Waite worked on the report sheets for one term from two London secondary schools for boys†. Each master was asked to judge the mental capacity of every boy he taught in five grades from very able to slow. Each boy was judged by four masters and the marks he might gain could vary between four and twenty, the higher numbers representing the better intelligence; these estimates

TABLE IV.

Grades of Intelligence.

	4 to 9	10 and 11	12 and 13	14 and 15	16 and over
Place in Examination	82	60	39	30	15
Place in Form (first school) ...	82	57	41	32	15
„ „ (second school) .	70	59	44	37	18

* *Biometrika*, Vol. VII. pp. 542—548. † *Biometrika*, Vol. VIII. pp. 79—93.

were then compared with the results of an examination held in all subjects in the one school and with the place in form gained for work done in school in both the secondary schools. We will give again the average place gained if there were 100 boys being considered.

This table shows again that when the master's estimate of the boy's intelligence is a low one the boy does badly in examination and in his class work, only reaching on the average to about eightieth in 100 boys, but that when the masters estimate the boy's intelligence as good, the place in examination and term work is about fifteenth in 100.

Mr W. H. Gilby considering boys in eight primary schools who were judged by 36 teachers all of whom had been in charge of their class for nearly twelve months found very similar results*. In this case the scale described before on p. 379 was used with one alteration, "*E*" the intelligent group being subdivided into *E* and *F* while *F* became *G*. The boys' intelligence as judged by their masters was compared with their place in an examination which was held by another man and with the percentage of marks gained by the boys in the head master's examination held the previous term. These are given in Table V below:

TABLE V.

Master's Estimate.

	B	*C*	*D*	*E*	*F*	*G*
Place in examination out of 100 boys	84	74	62	46	25	12
Percentage of marks gained in the head master's examination	43	54	61	70	79	85

The correlation coefficients given in Table VI show that there is a close association between the teacher's estimate and the school record.

TABLE VI.

	Teacher's estimate of Intelligence and Place in Examination		Teacher's estimate of Intelligence and Place in Form	
	Crude	Corrected for Age and Form	Crude	Corrected for Age and Form
Aberdeen	·72	·70	—	—
London Secondary Schools (1)	·69	·84	·62	·78
" " " (2)	—	—	·48	·61
Primary Schools †	·68	·68	—	—

Some masters were slightly influenced in their judgment of intelligence by the age and position in the school of the boy they were considering and allowance had to be made. It is interesting to notice that there is less agreement between

* *Biometrika*, Vol. VIII. pp. 94—108. † Mean of two values.

the teacher's estimate and the place in form in the second London Secondary school than in the first, but this can be accounted for, I think, by the fact that in the second school about a third of the boys had only been in the school for about three months and in the first school all boys had been in the school for at least six months and were consequently better known by their masters.

These investigations strengthened our faith in the estimates of intelligence and we decided to work out the correlation of intelligence in parents and adult children from data collected in the form of family schedules by Professor Pearson. In 1905 these forms had been issued asking for information about the health, intelligence, temperament, success in life, age at death and cause of death in families. Such schedules were formidable in size and the filled in forms came in so slowly that we are still asking for more. The data required were for a father and mother, their children, their parents, brothers and sisters, uncles and aunts, grandparents and cousins. It is I think very selected data with regard to intelligence; we should hesitate to give such a form to any one of say "slow" intelligence nor would such a person as a rule accept one! The result of this selection is that we have rather a large number of "very able" individuals, especially among the recorders, from the very fact that it was people of this type who were willing and able to give the required information. When we have more forms I think it will be as well to separate the person who filled up the form and his children from the grandparents and their children. Further there was a difficulty in the definition of intelligence. Only six classes were given and we found a tremendous heaping up in the intelligent class, the top but one, and Professor Pearson began to think that "slow intelligent" which to us meant average, meant to most people rather slow and that in the intelligent group we had many people included who were very little, if at all, above the average, so he divided the intelligent group into two parts "distinctly capable" and "fairly intelligent," but until we have more schedules in which this division is used there is some difficulty about grouping for working purposes. Tables VII and VIII give the results in rather a different form. We suppose that the intelligence of the community can be calculated in units called mentaces*, the larger the number of mentaces the more intelligent is the individual.

TABLE VII.

Father's Intelligence	Average number of Mentaces	Average Number of Mentaces in	
		Son	Daughter
Slow	269	348	260
Slow Intelligent ...	360	408	403
Fairly Intelligent ...	454	479	458
Distinctly Capable ...	554	497	472
Very Able	672	515	487
Mean	510	483	458

* By reduction to a normal scale of frequency: see *Biometrika*, Vol. III. p. 147.

The average number of mentaces of a father varies from 269 when he is "slow" to 672 when he is "very able." It will be seen that on the whole the "slow" and "slow intelligent" fathers do not have children as "slow" as themselves, but that their children have decidedly fewer mentaces than the average. The "distinctly capable" and "very able" fathers have children not so capable as themselves but they have a greater number of mentaces than the average mothers and children show the same thing but with even greater uniformity. The daughters of

TABLE VIII.

Mother's Intelligence	Average Number of Mentaces	Average Number of Mentaces in	
		Son	Daughter
Slow	266	385	351
Slow Intelligent ...	363	461	411
Fairly Intelligent ...	453	485	468
Distinctly Capable ...	566	501	482
Very Able	705	527	516
Mean	497	486	467

"slow" fathers had fewer mentaces than we should have expected but the daughters of "slow" mothers have done better than their mothers which is what we should expect from Galton's law of regression. The mean of the correlation coefficients for parent and child is ·44 which does not agree badly with what we find when we compare physical measurements in parent and child.

While these schedules were being collected the Binet-Simon tests had been extended for all age groups of normal children and the Intelligence Quotient began to be used as a measure of intelligence both abroad and in America. We had satisfied ourselves that the teacher's estimate of intelligence was on the average a sound one but there were two drawbacks to this method of judging intelligence which we fully recognized; the one was that it is very difficult to get a uniform standard over the whole country when you are dependent on definitions such as we have given of the grades of intelligence, and secondly you must have an intelligent and interested teacher. It is no good pretending that all teachers are equally intelligent, those that know the facts of their particular subject may be bad judges of other matters, but on the whole such teachers will not be sufficiently interested to undertake work of the nature I have described, which is unpaid and needs time and energy outside school hours, although as long as such estimates are made voluntarily there is very little danger under this second head. But the first difficulty is, I think, a real one, for though people in one town or district will use words with a certain meaning, travel 200 miles north and words such as "fair" begin to mean something different. Also if a man or woman has been teaching for long in any one school the average for that school may become his or her idea of "average" generally, and "fairly intelligent" in one school may

be only equal to "slow intelligent" in another. With these facts in mind the appearance of the Intelligence Quotient may be hailed with joy.

The use of mental tests in schools is still not very common in England though it seems to be gaining ground and we have to turn to America for the most extensive discussion of and work on the subject. For those who are ignorant of the meaning of the "Intelligence Quotient" I may explain that certain tests have been designed for each physical age of a child, which the average child of that age should be able to pass. The average child of five years' physical age would have a mental age of five years and its intelligence quotient would be the mental age divided by the physical age multiplied by 100 that is to say 100. If a child of five had a mental age of six years its intelligence quotient would be $(6 \div 5) \times 100$, i.e. 120, but if it failed at the five year old test and passed the four year old test only the intelligence quotient would be $(4 \div 5) \times 100$, i.e. 80, but a child can of course reach any mental age such as 5·2 or 5·8 years if it passes the five year old tests and some of the six year old ones. To the original Binet-Simon tests others have been added; the Stanford Revision is now more generally used in America and it has obvious advantages over the original Binet-Simon tests. It has to be remembered that Binet's work was pioneer work and after some experience it was found that his tests at the older ages were too difficult and there was a distinct fall in the intelligence quotient with the age of the subject. Many and varying tests have been used and I am not competent to express an opinion as to which test is the best to use or what method of scoring is best. On statistical grounds any test that has to make allowance for age is at a disadvantage. The actual tests employed under different systems seem very similar, but there are certain differences in the way in which the tests are given, the amount of help that may be contributed by the examiner and what percentage of tests for any year must be solved to attain the mental age of that year. In any mental tests the most important thing is to insure that one is using not a test of existing information, but a real test of intelligence, and those who have used these methods in America consider that they do test intelligence, not knowledge, and that children who have received no formal instruction may have a high mental age when competing with children who have been two or more years at school. Examples are given which individually are convincing but from a statistical standpoint one would like more extended experiments in this direction. Different people might consider that some tests were not entirely free from the objection I have named, that even if they did not test information they tested experience, nevertheless I believe a study of the tests shows that those who claim that they are testing intelligence and not knowledge are in the main justified in their assumption though I think myself there are some exceptions. On the whole one feels that if the instructions to the examiners are carried out faithfully the shy and nervous child will not be at a disadvantage, for the essential point is sympathy with the child and there are frequent warnings against confusing a shy child and instructions to the examiner to leave a test alone if the child seems bewildered and return to it later when confidence is restored. We all know what queer things will alarm certain small

children and examiners are warned to beware of this. In the test when three orders are given the subject is asked to place a key on a chair and it is suggested that some children will not touch a key and that if a child fails to start to execute the orders another test of the same sort not using a key shall be tried later. I cannot think that everyone would be an efficient examiner and I hold that the intelligence and sympathy that are necessary for the teacher who is going to estimate a child's intelligence adequately will be nearly if not quite as necessary in the examiner who is going to test a child's intelligence especially when dealing with young children. When the tests are standardized it seems to me the great advantage will be uniformity of definition, but I am not sure that the use of tests will displace entirely the teacher's judgment or that it would be wise that it should. These tests, as Binet says, have been specially designed to test general intelligence rather than some special faculty and such a faculty may not come to light if a Binet test is used exclusively. I should never suppose that any man or woman who had been with a child for a few hours or even for a few weeks would be a judge of that child's ability but a teacher who has seen a child in school and at play with its companions for a year ought, as a rule, if he or she is at all observant and sympathetic, to have a sound idea of that child's intelligence. There is not perfect agreement between the intelligence of the child as estimated by the teacher and as judged by the intelligence test, but both might be equally good estimates of intelligence judged from a rather different standpoint.

In the diagrams and tables so far given we have measured the strength of the inheritance of intelligence when that intelligence has been judged by some one who knows the children or adults concerned, now let us see whether the intelligence as judged by the intelligence quotient will yield like results.

In the *Journal of Delinquency*, Vol. IV. Dr Kate Gordon in testing the intelligence of children in three orphanages in California gave the intelligence of 91 pairs of siblings. Here we have very uniform environment and Professor Pearson found that the coefficient of correlation was ·51 which agreed very well with that found between brothers and sisters when the teachers estimated the intelligence; he discussed the question in *Biometrika*, Vol. XII. p. 367, and an examination of the diagram in that paper shows how closely the results for orphanage siblings resemble those for the English school children of Professor Pearson's own data. Dr Kate Gordon continued her investigations and in a Report to the State Board of Control of California on *The Influence of Heredity on Mental Ability* she gave the results of estimating the intelligence of 216 pairs of siblings. The Stanford Revision of the Binet-Simon scale was used, and the intelligence quotient was calculated. The examinations were carried out between July 1918 and September 1920, mainly on children in orphanages. The original intention of Dr Gordon had been to keep the pairs of siblings in each orphanage separate and to calculate the coefficients between siblings living under identical environment; she obtained results that varied from ·27 ± ·126 in one orphanage to ·69 ± ·086 in another, the latter result being obtained from 20 pairs of siblings and the former

from 26 pairs. The probable error of the difference is not three times that difference (·42 ± ·156) and we cannot therefore say that there is a significant difference between the pairs of siblings in these two orphanages but one feels very strongly that correlation coefficients obtained from such small numbers are essentially unsatisfactory.

The point that was very noticeable at the first reading was the correlation of ·61 which Dr Gordon found when considering 216 pairs of siblings, a distinctly higher correlation than we should have expected from the earlier examination of orphanage children which gave a value of ·51. As we were anxious to investigate the point a little further Dr Gordon most kindly sent us a copy of her original observations and a first examination led us to think that this high value of ·61 for the intelligence between siblings might be due to the diversity in race of the children considered which included Jews, Mexicans, and judging by the names, Italians, and other races. The expectation that we should find greater variability among these children than among the other orphanage children was not realized however as will be seen by an examination of the various constants for the 478 children who were examined.

TABLE IX.

	Later data	Earlier data
Mean Intelligence Quotient ...	90·48 ± ·47·	92·86 ± ·84
Standard Deviation	15·275 ± ·333	16·727 ± ·591
Coefficient of Variation ...	16·882 ± ·379	18·014 ± ·657
Correlation Coefficient... ...	·61 ± ·03	·51 ± ·05

If anything Dr Gordon's earlier series discussed by Professor Pearson in *Biometrika* shows a greater variability than this later series and a lower correlation and the source of the difference in the coefficient must be sought elsewhere; in itself the difference is not significant but ·61 is a higher value for the resemblance between brothers than we should expect to find.

Dr Gordon in considering families of which more than two members were tested used no child more than once, combining any one sibling with the next younger one. In this way families of three would yield only one pair of siblings, families of four and five two pairs and of six and seven three pairs and so on. As a check on Dr Gordon's result of ·61 we took all possible pairs, made the table symmetrical and then found a correlation coefficient for intelligence between siblings of ·467 ± ·026! The difference here is ·14 ± ·039 and is almost certainly significant. Whichever way you choose to pair the siblings both are samples of the same population and the results should not differ by more than three times the probable error, and if they do there is some anomaly somewhere. From the original data, using Dr Gordon's method of pairing, we found 219 pairs of siblings and a coefficient of ·593 ± ·030 which confirms generally Dr Gordon's result. There

was however a distinct difference in the mean intelligence quotient of the older and younger siblings which only showed itself when using this method of pairing; the mean intelligence quotient of the elder sibling was found to be 87·8 and of the younger 92·1. It is known that when the Binet-Simon test is used there is a lower average intelligence quotient for older children, but this difference in the quotient with age is not usually found when the Stanford Revision is employed; any such fact if it existed would account for the different coefficients of correlation found by the two methods of pairing. We then worked out the correlation coefficient between age and the intelligence quotient and found that it was $-\cdot310 \pm \cdot028$, that is to say the intelligence quotient decreased with age and the regression was $-1\cdot59$, which means that for every increase in one year of physical age the intelligence quotient was reduced roughly by $1\frac{1}{2}$. Diagram III, p. 391, for which I am much indebted to Miss I. McLearn*, shows very clearly a gradual decrease in intelligence with an increase in age. This seems to explain the variation in the coefficients found. When all pairs of siblings are taken and not only the pairs close to one another in age we may be decreasing the correlation automatically, and we must therefore correct for age before we can discuss the real correlation found for intelligence between siblings from the data. There are not enough pairs to make a separation of the sexes practicable, and such a division is not necessary here, for in orphanages where children of both sexes occur they do not differ significantly in the mean intelligence quotient. There are two possible methods of correcting for age. We can find the regression equation for the intelligence quotient on age, correct the quotient of each individual child and thus eliminate age. If we use this method and then proceed in the ordinary way with a symmetrical table we get a correlation coefficient of $\cdot540 \pm \cdot024$ which is a reasonable value in very good agreement with previous results found for the correlation in intelligence between brothers. Diagram IV, p. 392, shows that the regression is reasonably linear and that the points lie very evenly along the regression line. This method of correcting for age is not possible when broad categories are used for measuring intelligence; we must then use a partial correlation; we resolved to do this in the present instance as a check on the previous result; we further used Dr Gordon's method of pairing and then corrected for age to see how it compared. We decided first to use the method described by Professor Pearson in a paper *On Homotyposis in Homologous but Differentiated Organs*†, but to use this method the correlation between the growth in the elder brother's intelligence and the younger brother's age and between the younger brother's intelligence and the elder brother's age should be the same and this was only the case when all the tables were made symmetrical. From the figures given in Table X (p. 393) it will be seen that the differences are not really significant but in the two first columns we used partial correlation in the ordinary way.

* I owe the drawing of all the diagrams in this paper to Miss McLearn and would like to express here my appreciation of the value of her help. I have also cordially to thank Miss M. Moul for assistance in some of the laborious calculations.

† *Roy. Soc. Proc.* Vol. LXXI. p. 289.

PHYSICAL AGE & INTELLIGENCE QUOTIENT OF 478 CHILDREN IN CALIFORNIA

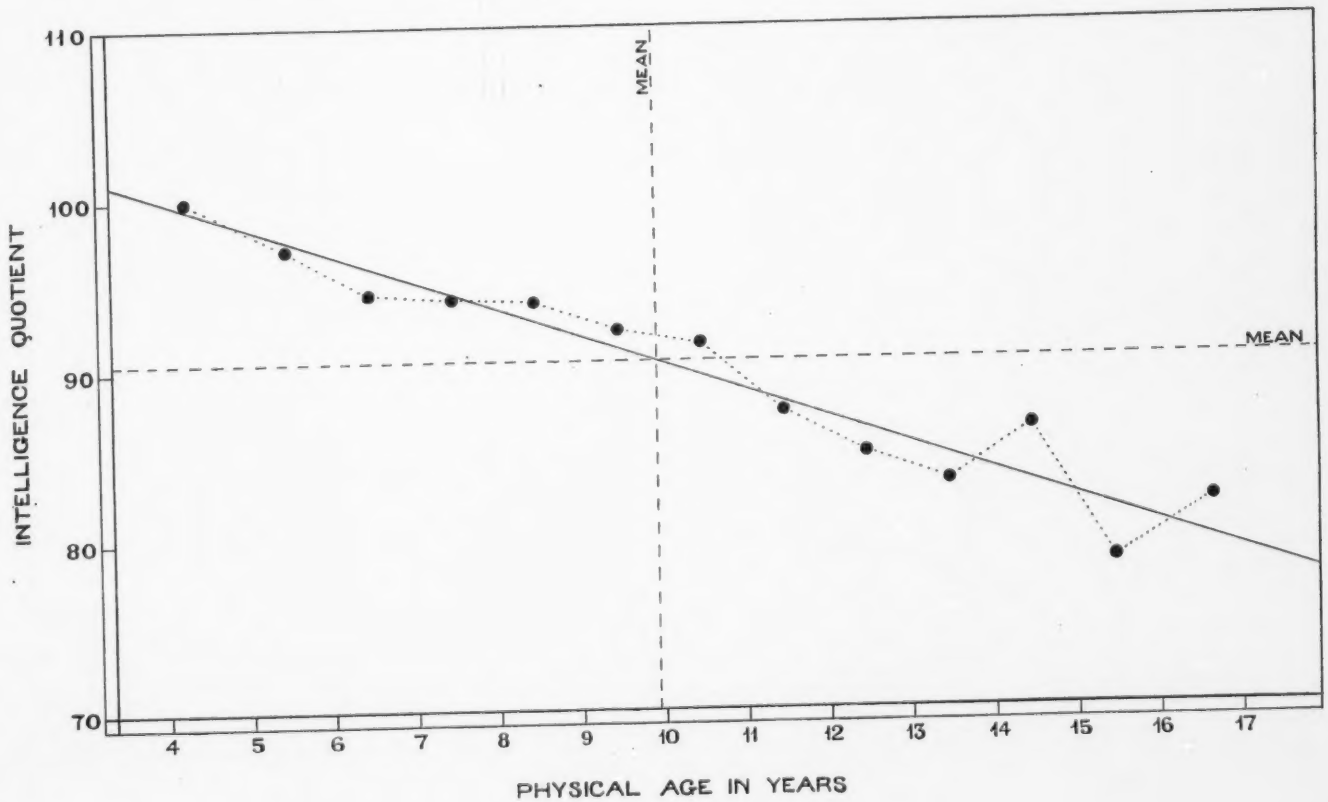


Diagram III.

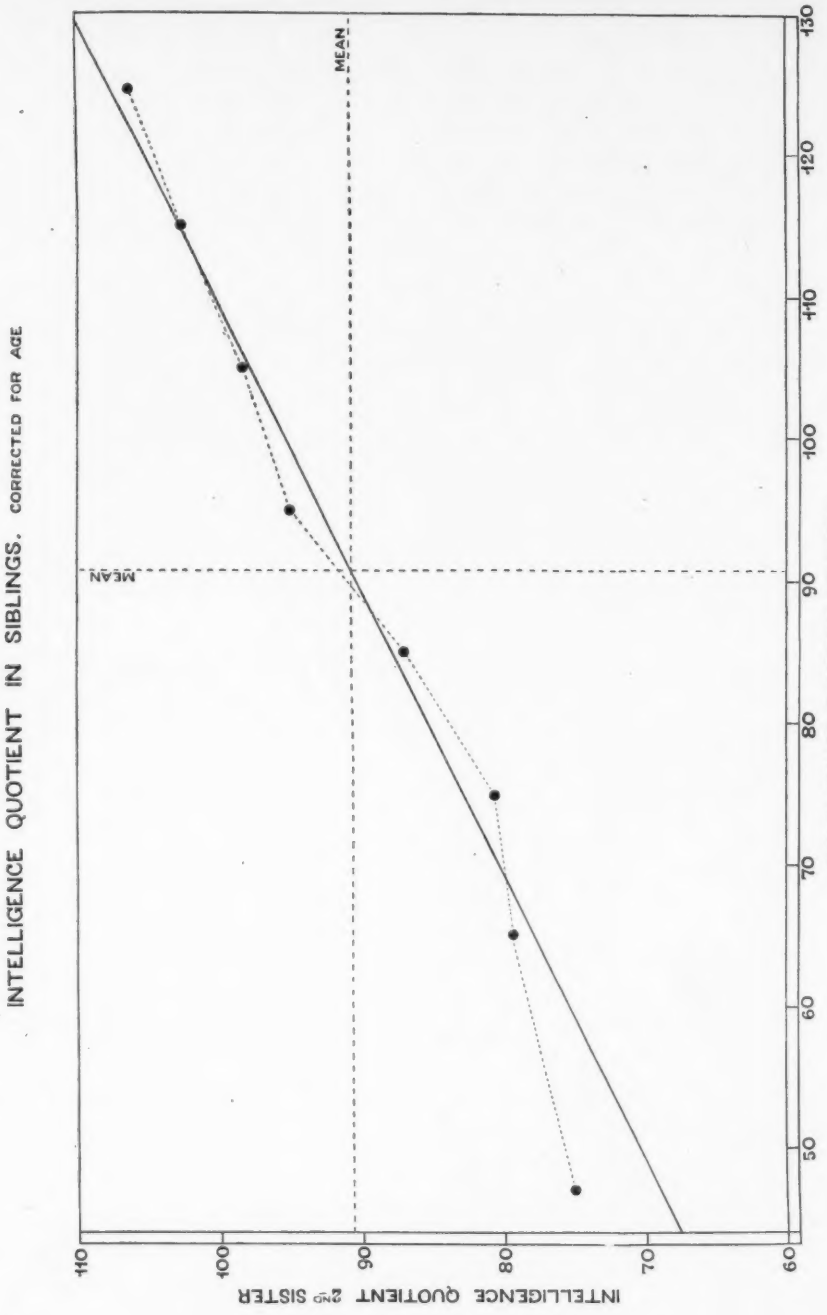

INTELLIGENCE QUOTIENT IN SIBLINGS. CORRECTED FOR AGE
MEAN
MEAN
INTELLIGENCE QUOTIENT 2ND SISTER
110
100
90
80
70
60
INTELLIGENCE QUOTIENT 1ST SISTER
130
120
110
100
90
80
70
60
50

Diagram IV.

It will be seen that when all pairs of siblings are considered the results agree very well with each other and with the correlation of ·540 obtained when we corrected each individual's intelligence quotient in accordance with age. Using Dr Gordon's method certain members of the family will constantly be omitted but even as it is there is no significant difference between ·58 and ·54 when the probable error in the first case is ·03 and in the second ·02. These values of ·54 agree very well with other results, namely those for adult brothers from the family records ·54 and for brothers at school ·52.

TABLE X.

	Dr Gordon's method of pairing	All pairs of siblings	All pairs of siblings and tables symmetrical
I. Q. of Elder and I. Q. of Younger...	·593 ± ·030	·534 ± ·024	·467 ± ·026
I. Q. of Elder and Age of Elder ...	− ·245 ± ·043	− ·280 ± ·031	− ·328 ± ·030
I. Q. of Elder and Age of Younger...	− ·189 ± ·044	− ·130 ± ·033	− ·022 ± ·034
I. Q. of Younger and Age of Elder...	− ·255 ± ·043	− ·183 ± ·033	− ·022 ± ·034
I. Q. of Younger and Age of Younger	− ·299 ± ·014	− ·266 ± ·031	− ·328 ± ·030
Age of Elder and Younger	·827 ± ·014	·702 ± ·017	·193 ± ·033
I. Q. of Elder and I. Q. of Younger for constant age of Elder and Younger	·578 ± ·030	·544 ± ·024	·532 ± ·024

The chief interest in this study of orphanage children lies I think in three directions. In the first place it shows that orphanage siblings resemble one another about as closely as children whose home conditions will vary in a more marked degree. In the second place it affords some evidence that whether the intelligence of children is measured by the intelligence quotient or by the estimate of teachers, the correlation between the intelligence of siblings is of the same order. It is obvious that in neither case have we the ideal conditions for making a dogmatic statement as the orphanage children were tested by the Stanford Revision and the school boys by the teachers, but I think the data collected add something to our knowledge of the hereditary factor in intelligence. Perhaps the chief interest really lies in the use of the intelligence quotient. One had hoped that the Stanford Revision had removed the necessity of allowing for age, which was one of the drawbacks to the Binet-Simon tests, but apparently it is not always the case. Here there is a well marked and steady fall in the quotient with age and we must either assume that there is something inadequate in the tests or that residence in an orphanage tends to decrease the intelligence of the inmates!

In any work dealing with an hereditary factor correction for age means a great deal of extra work and when dealing with environmental questions would add very much to one's labours especially if it were required to make some other factor constant. If this difficulty about age in connection with the use of tests cannot be conquered we shall begin to wonder whether a return to the results obtained by the now more old fashioned method of estimation has not much to recommend it. Professor Pearson showed in the paper in *Biometrika*, Vol. XII. before referred

to, how closely intelligence as obtained by estimate agreed with intelligence as measured by Stanford Binet-Simon tests and the labour is much reduced in dealing with any factor when age has been eliminated.

It may be suggested that since correction must be made for age, correlating the mental age of one sibling with another and making the physical age constant would be a better method of proceeding since it would save calculating the intelligence quotient, but here again there is a difficulty. It was found that there was greater overlapping in the scores of the older children than in the scores of the younger ones* and that one year in physical growth did not correspond to one year in mental growth. For instance, taking a standard test for five year old children, it was found that about twenty-three per cent. more six year old than five year old children passed the tests, but if a standard test for ten year old children was taken it was not twenty-three per cent. more of the eleven year old children but twenty-three per cent. more of the twelve year old who passed the test. At present these tests are still partly in the experimental stage and such an anomaly as that to which I have referred might be removed, but until it has been, we must continue to use the intelligence quotient. At present whatever tests are used an investigation into the relationship between age and the quotient must be undertaken before it will be safe to draw any conclusions of either an hereditary or environmental nature. In many instances I have seen the Stanford Revision used when there was no correlation with age, but certainly in the data here examined such a correlation existed and though correction for age did not make a very great difference it showed that the question of age cannot at present be neglected when considering intelligence as measured by the intelligence quotient.

From Dr Gordon's data collected in orphanages we found that when intelligence is measured by the Stanford Revision tests the resemblance between siblings is about ·54, while from data collected by Professor Pearson where the environment presumably differs more widely and when the intelligence is estimated by the teachers the resemblance between brothers is ·52 at school and ·54 for adult brothers when the judgment is that of a relative, and we feel that the inheritance of intelligence is a fact very little if at all affected by environment. It might be argued however that in making this statement we were comparing facts that were not strictly comparable, since we had no evidence that if the orphanage children had been judged by the teachers we might not have found a much higher association between the intelligence of siblings than we actually found in the school children, and that if the Charterhouse boys and the brothers in the family schedules had been tested by Binet-Simon tests there might have been less association found than there was when they were judged by place in school or by the estimate of their relatives. What we want are pairs of siblings who have been judged in both ways, by test and by the teacher's estimate, and if the correlation between intelligence in siblings comes out the same in both cases we shall feel that our conclusions are justified. Owing to the energy and kindness of Dr Drinkwater of Wrexham we have recently received some material in which both methods of

* *Journal of Educational Psychology*, Vol. XII. January 1921, p. 3 et seq.

ascertaining intelligence have been applied to the same children. The numbers of pairs of brothers and sisters are however very small and owing to a difference in the intelligence of boys and girls we cannot combine the sexes here, but the material is, as far as I know, the first of this kind collected for brothers and sisters and is therefore valuable*.

Our most grateful thanks are due to Dr Drinkwater and to the head teachers of the schools concerned who made the testing possible, and to all those teachers who estimated the intelligence of the children. The first examination was of pairs of brothers and sisters in a high class school in Wrexham, the best school in the district in which the parents formed the aristocracy of the working classes in the town. Dr Drinkwater first used the Binet-Simon tests but found that for children of ten years and over the tests were too difficult and he then used the Stanford Revision on all children over ten and on all children that he examined subsequently. Dr Drinkwater re-tested some of the younger children by the Stanford Revision and found that the intelligence quotient obtained by the Binet-Simon method was confirmed by the Stanford Revision, but one feels that it is not very satisfactory to have two different tests used in the same school and sometimes on members of the same family, and we shall see presently that the results in this first school are more irregular than in the second school where the method employed was uniform throughout. The teacher's estimate was also obtained for each child quite independently of the tests, which were carried out by Dr Drinkwater himself. The second school was of a very different type and the children were inferior in every respect to those who were found in the first school, home housing conditions were worse and the occupations of the fathers were less skilled. The average intelligence quotient differed in the two schools and owing to the different tests used in the better class school we preferred at first to keep the two schools separate, though we combined them later; such a separation of the schools has the disadvantage that we are dealing with comparatively few cases. Table XI gives the average intelligence quotient for the two schools for boys and girls, and for convenience we shall call the schools *A* and *B*, *A* being the better class school which was examined first.

TABLE XI.

	Average Intelligence Quotient	
	A	*B*
Boys ...	101·4	94·0
Girls ...	97·2	88·7

It will be seen that in both schools the average intelligence of the girls is lower than that of the boys and that the intelligence in the inferior type of school is lower

* Dr P. Stocks has collected similar data for individuals and gave some of his results in a paper read before the Society of Biometricians and Mathematical Statisticians, and the correlations he found between the intelligence quotient and the teacher's estimate of intelligence agree very well with the results obtained from Dr Drinkwater's data.

than in the better. These conclusions are confirmed by the teacher's estimate. Seven groups were used for grading the intelligence, *A*, *B*, *C*, *D*, *E*, *F* and *G*, but in all cases we grouped *A* and *B* together as "very dull" and *F* and *G* together as "distinctly capable," *C* can be defined as "dull," *D* as "rather slow" and *E* as "intelligent." Diagrams V—VIII on p. 397 will show that the average intelligence of boys in school *A* is in the "intelligent" group, though fairly close to the boundary of "slow," but that the average for the girls is in the "slow" group. In school *B*, the average for the boys, according to the teacher's estimate, is almost on the boundary line of "slow" and "intelligent," and for girls on the boundary between "slow" and "dull." We shall return to this difference in intelligence in the sexes later; we did not find it in the orphanage data collected by Dr Gordon, nor has it generally been found in America when boys and girls under 15 years are considered.

We first found in both schools the correlation between the teacher's estimate of intelligence and the intelligence quotient. The diagrams to which we have already referred on p. 397 will show that there is a well-marked association and that the regression may be considered linear, though among "very dull" boys we find a lower intelligence quotient than we should expect, but the mean is based on only ten cases. We found the following values for the correlation ratio between the teacher's estimate of intelligence and the intelligence quotient:

	School *A*	School *B*
Boys ...	·662 ± ·029	·581 ± ·030
Girls ...	·617 ± ·026	·789 ± ·021

This does not show anything like perfect correlation but only a well-marked association. In one or two cases where Dr Drinkwater noticed a decided difference between the teacher's estimate and the intelligence quotient he re-tested but without finding much change. One would hardly expect perfect correlation between the two and it is quite possible that both are equally good measures of intelligence from a different standpoint.

The next point to be discussed is the relationship between age and intelligence; it will be remembered that Dr Drinkwater used the Stanford Revision test after experimenting with the Binet-Simon in an effort to obtain a measure of intelligence independent of age. As far as the boys are concerned he was successful and there is no association between the intelligence quotient and age in either school, but this is not the case with the girls, for we find a correlation coefficient of $-·225 \pm ·040$ between age and the intelligence quotient in school *A*, and of $-·297 \pm ·049$ in school *B*. Both these coefficients are significant, but since no decrease in the intelligence quotient is found with age among the boys we were at once inclined to attribute it, not to the tests adopted, but to some other factor and this idea receives some support from the fact that there is a correlation between age and the teacher's estimate of intelligence for girls. For girls the correlation coefficient is $-·146 \pm ·041$ (assuming a normal distribution for intelligence) in school *A* and $-·241 \pm ·051$ in school *B*, but there is no decrease in intelligence as measured

INTELLIGENCE QUOTIENT & TEACHER'S ESTIMATE

Dr Drinkwater's data, earlier Series

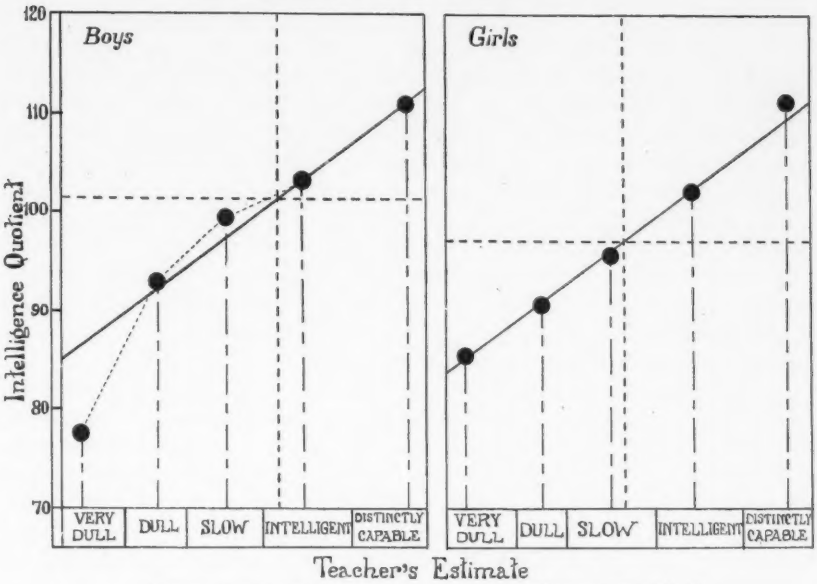


Diagram V. School *A*. Diagram VI. School *A*.

INTELLIGENCE QUOTIENT & TEACHER'S ESTIMATE

Dr Drinkwater's data, later Series

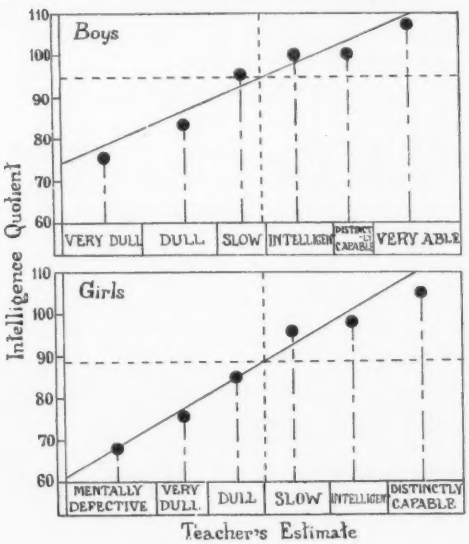


Diagram VII. School *B*.

Diagram VIII. School *B*.

by the teacher or by the intelligence quotient with the age of the boys, all the coefficients found being positive and none significant. This seems to indicate that for some reason the older girls in these schools are less intelligent. If it occurred in boys as well as in girls and began suddenly at age 11 (in school *B* there is a tendency for this to happen among the girls, but the numbers are too few to be sure on this point) we might attribute it to the transference of the brighter children to secondary schools. Under our present educational system this takes place at about the age of 11 years, and there is evidence in the notes received from the school that the older children in some families are at the county school, but we should expect this transference to apply to boys as well as girls. There seemed only two possibilities which would account for the difference between boys and girls and neither is really confirmed by further information received from Dr Drinkwater. First that for some reason the brighter boys in Wrexham do not enter secondary schools but go straight to work, or that there were so many more places for boys than girls at secondary schools that many boys who were only just "above the average" were transferred, while only the brightest girls went on, but this second suggestion will not really account for the entire absence of any decrease of intelligence with age in the boys, because the brightest boys would certainly be transferred first. The suggestion that the brightest boys go straight to work is not confirmed by further enquiries carried out by Dr Drinkwater as the following extract from the letter of the head master of school *A* will show. "With regard to the falling off of the I. Q. in children of 11 and upwards* a very decided factor in the case of my own school is the fact that such a big number of my brightest boys go to the secondary school at 11 and 12. The last two years I lost over 30 each year and this out of about 120 which is a large percentage. Another cause might be the employment morning and evening of a number of my bigger boys." With reference to the girls in school *B* the head mistress reports that 15 girls were transferred to a county school and 13 to a convent school and adds, "Some of our girls from 11 years upwards have to work very hard at home, especially when the mothers are ill, etc. I cannot help thinking that the result of the war stress is being shown now as I have a number of retarded children."

The head mistress of school *A* writes, "During the three years prior to your inspection 48 girls, who would normally have been in the senior classes at the time of your inspection, passed on to the secondary school through Scholarship or Free Place. Free Places are, as you know, awarded to all scholars who attain a certain standard of work. In addition another 6 left for various secondary schools as paying pupils. You will see therefore that our brightest material is passed out of school between 11 and 12 years, consequently only inferior material is left. As a rule this material comes from socially inferior homes, and the scholars are kept away a good deal for work in the homes. There is another factor which I think helps to account for the low quality of our senior material. These girls have grown up during the upset of war time—throughout the 7 years of their attendance national

* As we have said this "falling off" does not occur among boys but it would appear from this letter that it should do!

conditions have been abnormal, and in many instances homes have reflected this abnormality—there has been less discipline, and this fact may have reacted on the children*. The aspect of the senior classes of primary schools has changed very considerably of late years—we have practically only the residue of children who are not able to benefit by higher education—entrance to secondary schools has become increasingly easy, and there are so many free places, that a much higher proportion of pupils passes on now than formerly."

It is quite clear from these letters that the more intelligent girls do go on to secondary schools, which accounts for the lower intelligence quotient found among the older girls left at school, but we are just as much in the dark as to why there is no trace of this among the boys. A much larger proportion of boys than girls go on to secondary schools. "Over 30 boys" in one year in school *A* as compared with only 16 girls from the same school†, and apparently there are more girls in that school than boys. We have only before us the cases in which more than one of a family is considered, but from these data we find 188 girls under 11 years as compared with 132 boys, so that we may infer I think that the boys more frequently move on to secondary schools than the girls, but even if this be so we have not accounted for the absence of any decrease in intelligence among the older boys in these schools. It has been suggested that the oncoming of puberty which is earlier in girls than in boys means a diminution in intelligence between the ages of 11 and 14, but such an explanation has received no confirmation from work done in America, where retardation occurs after 14 and not before when Stanford Revision tests are used‡.

Table XII gives the crude correlation coefficients, that is to say, uncorrected for age, between the different types of sibship in the two schools, using in each case two measures of intelligence, the intelligence quotient and the teacher's estimate.

TABLE XII.

	School *A*		School *B*	
	Intelligence Quotient	Teacher's Estimate	Intelligence Quotient	Teacher's Estimate
Brother—Brother ...	·669 ± ·053	·768 ± ·040	·394 ± ·055	·351 ± ·057
Sister—Sister ...	·272 ± ·060	·450 ± ·052	·461 ± ·065	·460 ± ·065
Brother—Sister ...	·326 ± ·044	·543 ± ·036	·449 ± ·036	·466 ± ·035

Correction for age makes only slight changes and from this table it is obvious that the results found in school *A* are very irregular. If we consider only school *B*

* A good deal of this letter seems to show that the school mistress reckoned knowledge as a factor of intelligence! This would weaken the correlation of the teacher's estimate with the Binet-Simon tests record.

† It would be 18 per year if we included those who went on as paying pupils.

[‡ There are distinct traces of it in the change of psychical characters with age in my school data and this remark applies much more to girls than boys. K. P.]

we find that the resemblance in intelligence among siblings is practically the same whether we judge that intelligence by the tests or by the teacher's estimate and such a result gives one great confidence in the inheritance coefficient as worked out solely on the teacher's estimate in the past. For all pairs of siblings it is lower than we should expect but it must be remembered that we are dealing with selected data, with a school in which the pupils rarely come from better class homes, that therefore the variation will not be very marked and consequently the correlation coefficients will not be as high as if we were dealing with more variable material. The coefficients of variation in this school do not differ significantly from those found for orphanage children when the whole series was taken; here they are 16·3 for boys and 16·9 for girls and among orphanage children the coefficient of variation was 16·9 for both sexes taken together. In school *A* on the other hand the resemblance in intelligence between sisters and between brothers and sisters seems to differ according to the measure of intelligence used. The differences with the probable errors are

Sister-Sister	...	·178 ± ·067,
Brother-Sister	...	·217 ± ·057.

In the first case the difference is not three times the probable error and may or may not be significant, but in the second case the difference does exceed three times the probable error and must, I think, be regarded as significant. The value found for the resemblance between sisters when the intelligence quotient is used is very low absolutely and relatively as compared with the other school and though the coefficient of variation has sunk to 13·48 it seems a hardly sufficient diminution in variability to account for the low value. For brothers the correlation is much higher than in school *B* but as the number of pairs of brothers is only 49 we cannot expect results that are likely to be of much value. Comparing the two measures of intelligence there is fair agreement. The resemblance between pairs of sisters and pairs of brothers and sisters when the teacher's estimate is used is much the same in both schools but we certainly cannot say the same for the intelligence quotient and correction for age makes very little difference. Can these irregularities be due to the use of two different tests on some of the younger children in this particular school?

Diagrams IX—XIII show the regression lines for each type of sibship and for each school, the unbroken line giving the regression line when the intelligence is measured by the intelligence quotient and the broken line when it is estimated by the teacher. It must be remembered that in every case means are based on a very few cases which causes many irregularities. The general trend of the two regression lines in school *B* is very similar but as we should expect the agreement is not so good in school *A*. Possibly a brother-brother diagram in school *A* would show a better result but when only 49 pairs of brothers are available the drawing of a diagram seems waste of time. We must now correct the girls' intelligence for age since there is a quite significant correlation between age and intelligence. There is no need to apply this correction to the boys since the correlations are

·050 ± ·051 in school *A* and ·069 ± ·046 in school *B* when intelligence is estimated by the intelligence quotient and ·031 ± ·049 and ·103 ± ·045 when the intelligence is estimated by the teachers. It will be obvious that none of these are significant correlations and all are positive which emphasizes the difference from the girls where in all four cases we found negative significant correlations between their age and intelligence. The first method used for correcting for age was applied to the

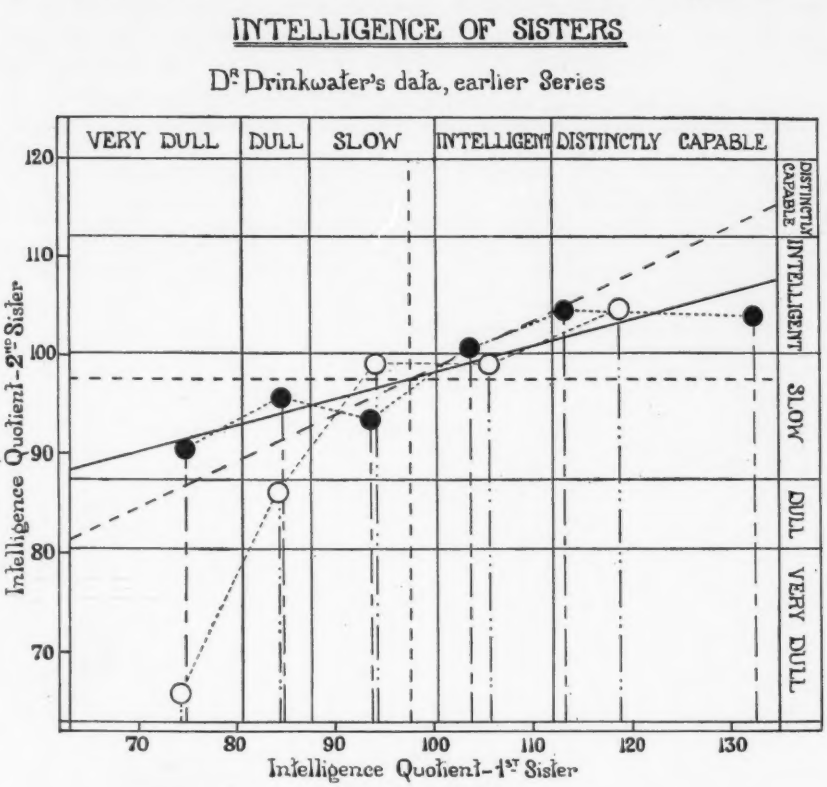


Diagram IX. School *A*.

intelligence quotient only; the regression of intelligence on age was found and the required correction made in each individual case with the following results.

		School *A*	School *B*
Sister-Sister	...	·174 ± ·062	·507 ± ·061
Brother-Sister	...	·296 ± ·045	·445 ± ·036

It will be seen that only slight changes have been made, but in both cases in school *A* the coefficients of correlation are even lower than they were before and

that between sisters occurring alone would hardly be significant. We will now use partial correlation and find for both teacher's estimate and intelligence quotient the correlations when age is constant. Working with broad categories we can find the contingency coefficient (C_2) or, assuming a normal distribution for intelligence, we can find "r." We have worked out the tables in both ways and give the two results; correcting for age makes very little difference, in fact, taking into consideration the probable error, we have made no significant change in our coefficients and it may be asked why we took the trouble to make the corrections, but until the whole thing is worked out it is not possible to foresee the result. When any

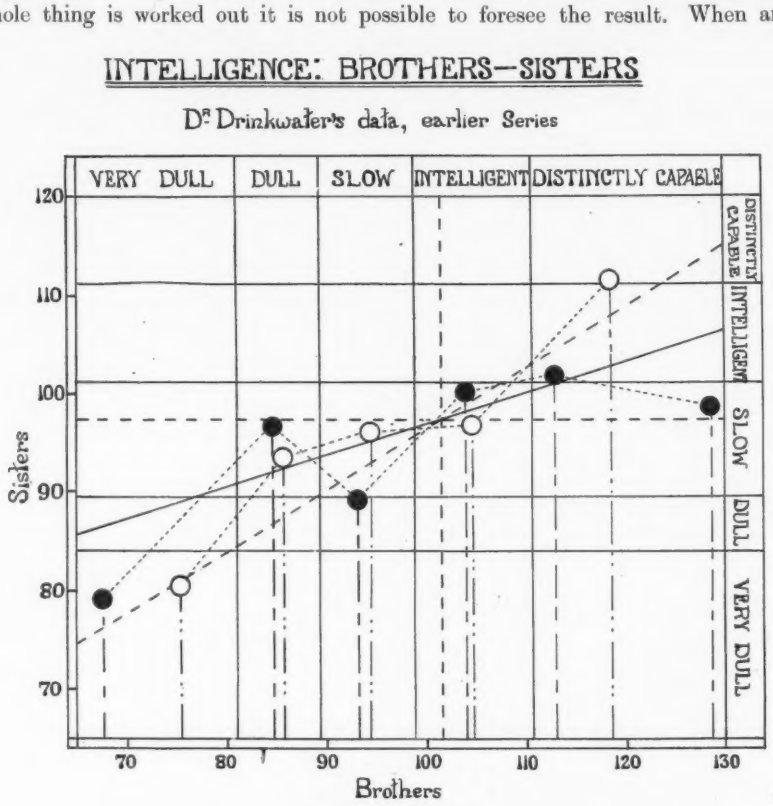


Diagram X. School *A*.

significant correlation exists between age and some character the correction for age must be made. When the changes are so slight it seems needless to repeat the comments made on the crude table. The probable errors are large and, taking this fact into account, School *B* seems to give results which we might anticipate and it makes very little difference whether the estimate of intelligence is by a test or by the teacher, but in School *A* the anomalies remain as marked as before.

INTELLIGENCE OF BROTHERS

Dr Drinkwater's data, later Series

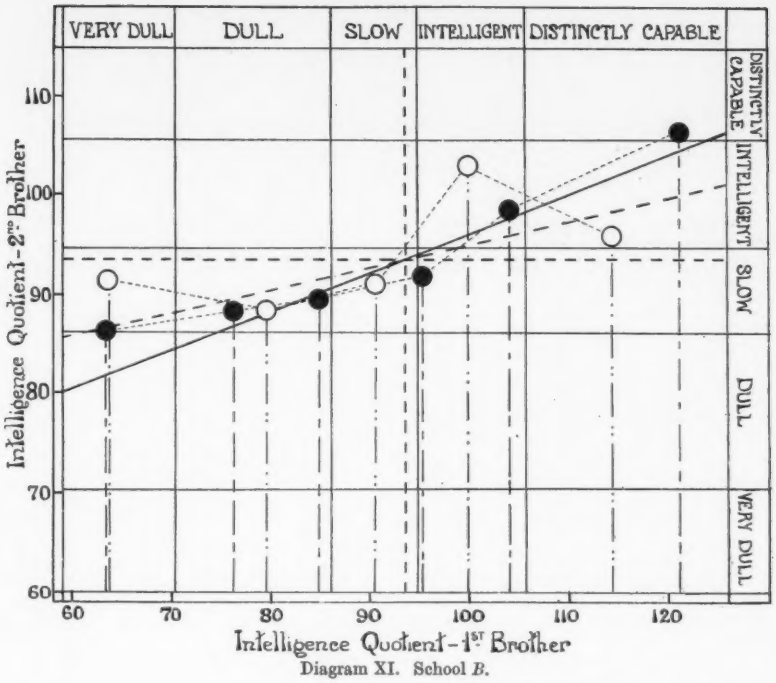


Diagram XI. School *B*.

INTELLIGENCE OF SISTERS

Dr Drinkwater's data, second Series

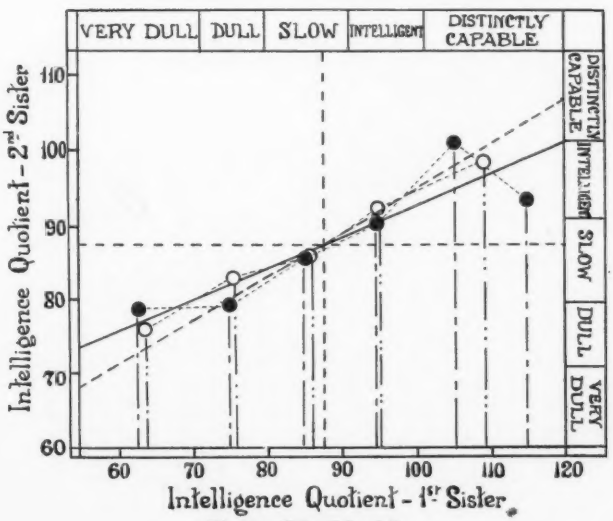


Diagram XII. School *B*.

There seems a definite reason why the sexes should not be combined, but no really cogent reason* for not combining the schools except the fact of the change of test in School *A* among the younger children, so we decided to put together the pairs of siblings from the two schools and see whether larger numbers would give

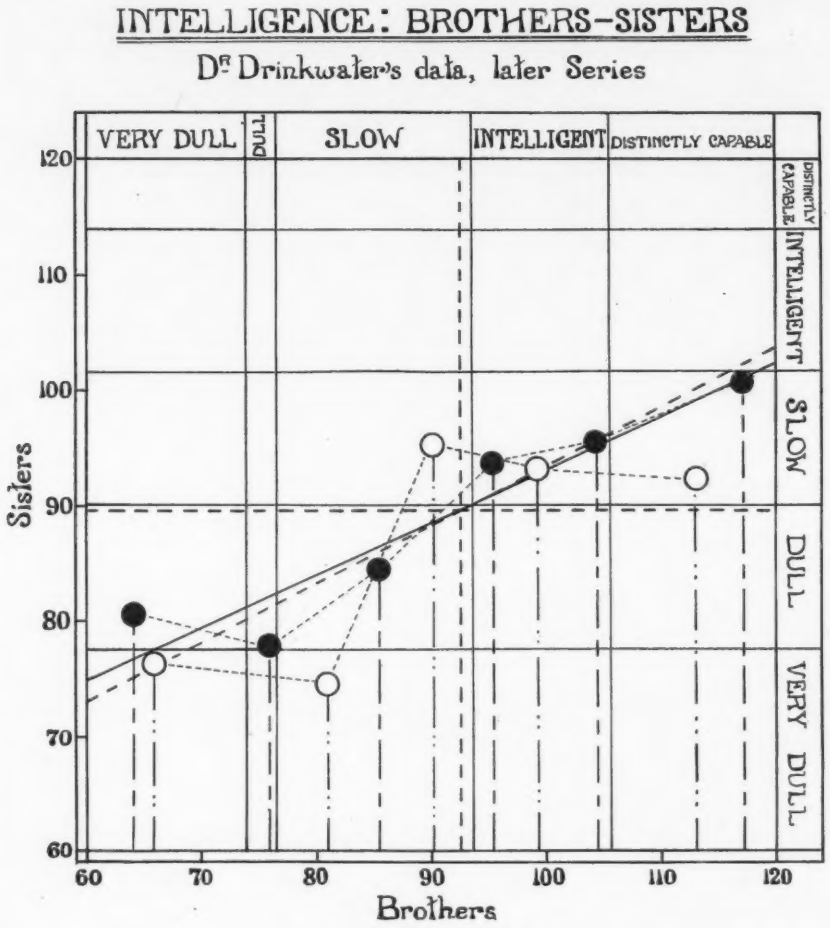


Diagram XIII. School *B*.

us greater uniformity of result. It is not getting over the differences between the schools but is perhaps worth while considering. First we correlated age with teacher's estimate of intelligence and then with the intelligence quotient and found

* There is a possible reason for combining them in that the two schools have really selected each a portion of the general population.

TABLE XIII.

Inheritance of Intelligence.

		Estimated by the Intelligence Quotient		Estimated by the Teacher	
		School *A*	School *B*	School *A*	School *B*
Pairs of Sisters	Intelligence of Elder and of Younger	·272 ± ·060	·461 ± ·065	·450 ± ·052	·460 ± ·065
	" " " Age of Elder ...	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·041	− ·241 ± ·051
	" " " Age of Younger ...	− ·254 ± ·060	− ·226 ± ·079	− ·198 ± ·062	− ·254 ± ·078
	Intelligence of Younger and Age of Elder ...	− ·270 ± ·060	* − ·060 ± ·083	− ·182 ± ·062	− ·204 ± ·080
	" " " Age of Younger	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·041	− ·241 ± ·051
	Age of Elder and Younger	·610 ± ·040	·509 ± ·062	·610 ± ·040	·509 ± ·062
	Intelligence of Sisters for constant age { using r	·221 ± ·061	·462 ± ·065	·470 ± ·050	·411 ± ·067
	{ using C_2	—	—	·603 ± ·042	·451 ± ·066
Pairs of Brother-Sister	Intelligence of Brother and of Sister... ...	·326 ± ·044	·449 ± ·036	·543 ± ·036	·466 ± ·035
	" " " Age of Brother ...	·050 ± ·049	·069 ± ·046	·031 ± ·049	·103 ± ·045
	" " " Age of Sister ...	− ·032 ± ·049	− ·102 ± ·045	·041 ± ·049	− ·152 ± ·040
	Intelligence of Sister and Age of Brother ...	− ·042 ± ·049	·207 ± ·043	− ·148 ± ·048	·096 ± ·045
	" " " Age of Sister ...	− ·221 ± ·040	− ·297 ± ·049	− ·146 ± ·048	− ·297 ± ·049
	Age of Brother and Sister	·151 ± ·048	·009 ± ·045	·151 ± ·048	·009 ± ·045
	Intelligence of Brother-Sister for constant age { using r	·329 ± ·044	·438 ± ·036	·564 ± ·035	·440 ± ·037
	{ using C_2	—	—	·520 ± ·036	·429 ± ·037
	Pairs of Brothers	·669 ± ·053	·394 ± ·055	† ·768 ± ·040	† ·351 ± ·057

again a negative result for girls − ·266 ± ·036, when the estimate was the intelligence quotient and − ·309 ± ·035, when the estimate was that of the teacher, while for boys it was ·008 ± ·034 and ·060 ± ·034 respectively for the two estimates of intelligence. Again in this case we must correct the tables in which the girls appear for age and we get the following.

TABLE XIV.

	Intelligence in Siblings judged by	
	Intelligence Quotient	Teacher's Estimate
Brother-Brother	·489 ± ·041	·486 ± ·041
Sister-Sister	·425 ± ·041	·580 ± ·034
Sister-Sister corrected for age ...	·384 ± ·042	·540 ± ·036
Brother-Sister	·529 ± ·024	·486 ± ·025
Brother-Sister corrected for age ...	·532 ± ·024	·499 ± ·025

* Re-checked with the same result, but note the large P.E.

† Using C_2.

Here the results are more uniform, we have nothing so strange as the value found for the resemblance in the intelligence quotient between sisters in School *A*, and except for sisters the two estimates of intelligence agree very well. It is as well to notice that on the whole the resemblance between siblings when the teacher's estimate is taken is lower than when the Stanford Revision test is used, from which we may fairly argue, I think, that the results already obtained for inheritance of intelligence by biometricians are not over-estimated even though use has been made of the teacher's judgment rather than of some numerical test *. The resemblance between brothers and sisters and between parent and child is very similar in value to the resemblance found for physical characters and we must now see whether there is anything equal to heredity in determining the grade of intelligence. The Americans have done some work on this point to which I shall refer in a moment but there is room for much more. In our own laboratory Dr Heron took the intelligence of children in the London elementary schools and compared it with the physical condition of the children, with stunted growth and with bad home condition as measured by cleanliness. Since the estimates varied in every school all were kept separate and the results given in Table XV represent the means of many observations, negative in some schools, positive in others.

TABLE XV.

Intelligence and	Boys	Girls
Height ...	·10	·07
Weight ...	·06	·03
Cleanliness...	·14	·07
Nutrition ...	·01	·08
Glands ...	·09	·08
Tonsils ...	– ·01	·11
Teeth ...	·08	·09

These coefficients are all small, many not significant, the highest is between cleanliness and intelligence in boys, which is ·14, which holds the same position in the scale of intelligence that 14 per cent. holds in the scale of percentages. Such a coefficient though significant in itself may be only a secondary effect of intelligence; if the more intelligent parents and the better type generally keep their children cleaner than the less intelligent parents there should be a correlation between intelligence and cleanliness in the child, but ignoring this possibility we can see that the correlation coefficients given in Table XV are all small and not comparable in size to those found for the hereditary influence.

Other data that we have are from Edinburgh where the results found are given in Table XVI. Here the numbers dealt with are much smaller and none of the results found are significant except those between the intelligence of the children

* It is impossible to consider the data here dealt with as final, not only are the numbers inadequate, but there is a want of that smoothness, which almost invariably will be found when judgments are given by specially trained examiners dealing with adequate numbers.

and the economic condition of the home, and unless one is to suppose no connection between good homes and a rather better type of intelligence in the parents such a relationship may be a secondary effect of the hereditary nature of intelligence. Some of the coefficients are negative and some positive and none of them mean anything—there is no association between the moral and physical condition of the parents and the intelligence of their children, and only a very slight one between the economic condition of the home and the intelligence of the children, which may be an indirect result of heredity.

TABLE XVI.

	Boys	Girls
Intelligence and drinking of the father...	−·07	·05
,, ,, ,, mother.	·05	−·09
Morality of the parents	−·07	·03
Physical condition of the parents ...	−·04	·06
Economic condition of the home ...	·10	·16
Overcrowding	·02	·04

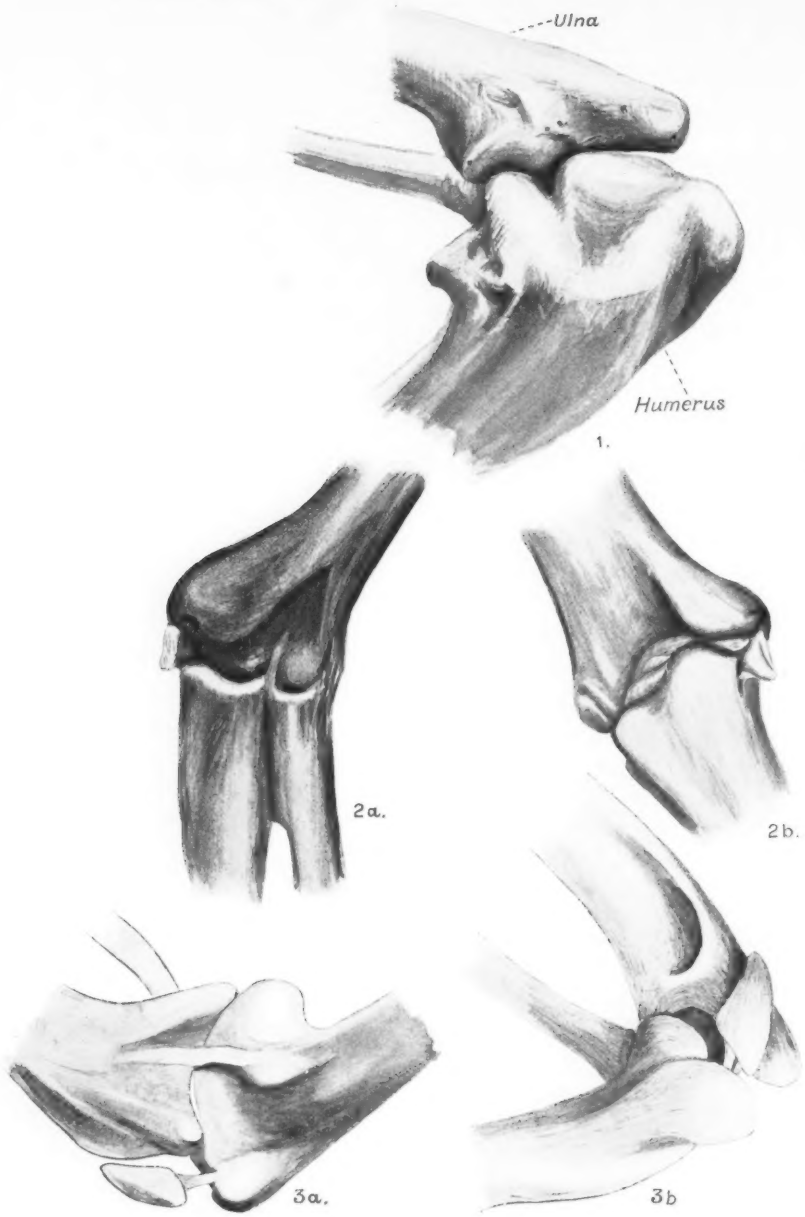
As far as we can see from all the material we have examined the environment matters little compared with the hereditary factor. Here we are still working on the teacher's estimate of intelligence, but there seems no reason to suppose that the intelligence quotient will yield different results for environment since it did not give us very different results for heredity, but before long we ought to be able to test children in Great Britain and work out the influence of environment on the intelligence quotient. Some work in this direction has been done in America already and Terman* considers that the culture of the home, apart from heredity, has little influence and that of the dozen or more children who have been removed from an unsatisfactory to a satisfactory home environment not one showed an improvement in his intelligence quotient. "A dozen or more" cases are not enough but if every case of this kind were noted, I mean wherever there was a change of environment, we should in time get some real evidence on the point.

In America also much work has been done showing the persistence of the intelligence quotient from year to year in the same child. Re-tests were given to 315 children and the correlation between one test and a later one was ·93. Some of these children were re-tested more than once and all the cases were used. There were 33 examiners and in 72 per cent. of cases the earlier and later tests were given by different examiners. In 86 cases the re-test was at less than a year's interval, in 138 cases between 1 and 3 years, in 85 between 3 and 5 years and in 127 there was more than 5 years between the tests and Terman, on this basis, considers that, apart from minor fluctuations due to temporary factors, the feeble minded remain feeble minded in spite of special schools, the dull remain

* Lewis M. Terman, *The Intelligence of School Children*, London, 1921.

dull, the average remain average and the superior remain superior. This does not by any means imply that education is unnecessary, it sharpens the tools and provides the material for the brain to use, and the keener the instrument the more the brain can achieve, but the efficiency of the output is I think limited by the power of the brain that produces it. In the midst of many new experiments in education the use of re-tests would quickly prove whether intelligence can be increased by any or all of these new systems. Even if it were proved that no real increase in intelligence followed a change in educational environment it does not mean that we should remain in our old ways. New methods may improve the edge of the razor without changing its metal, but in these days it is important, I think, to distinguish between the possibility of increasing intelligence and the possibility of giving an existing intelligence help in achieving success. Education in even the ideal form may fail in the former case and succeed in the latter but they are not one and the same thing and it is no use to have a perfect system of education if the brains that should profit by it are decreasing in power, if men and women of intelligence are leaving fewer and fewer children to inherit their ability.

All men are not born equal. I must quote Francis Galton, for after all he still remains one of our chief authorities on inheritance—"I acknowledge freely the great power of education and social influences in developing the active power of the mind, just as I acknowledge the effect of use in developing the muscles of a blacksmith's arm, and no further. Let the blacksmith labour as he will, he will find there are certain feats beyond his power that are well within the strength of a man of herculean make, even although the latter may have led a sedentary life.... In running, in rowing, in walking, and in every other form of physical exertion, there is a definite limit to the muscular powers of every man, which he cannot by education or exertion pass. This is precisely analogous to the experience that every student has had of the working of his mental powers." To each of us, a limit is set, a limit as far as one can see at present due to heredity rather than to opportunity and to the intelligence of our parents and ancestors rather than to the educational system under which we were reared.



1. Ulna of Eagle, left wing, ventral aspect.

2 a. } White-tailed Eagle, left wing, Cartilaginous Sesamoid of Elbow-Joint in two aspects. 2 a anterior
2 b. } and 2 b posterior.

3 a. } House Sparrow Elbow-Joint, Posterior and Ventral views (×8) to show Ulnar Sesamoid.
3 b. }

Addendum to a "Memoir on the Sesamoids of the Knee-Joint."

Biometrika, Vol. XIII. pp. 133–175 and pp. 350–400.

On p. 387 of the above memoir reference was made to Owen's statement that in certain birds there is an ossiculum belonging to the ulna which Owen states is "essentially the separated olecranon of that bone." This detached sesamoid, which has been termed the "ulnar patella," is stated by Owen to occur in many of the *Raptores* and in the Swifts, while in the Penguins it is double. The importance of the matter for the above paper lay in the association as homologous of the olecranon with the cnemial crest of the tibia, and the possibility of the sesamoids of the elbow and knee-joints arising from a partial break up of these appendages.

I have examined the elbow-joint of a number of *Raptores* without much success in the discovery of ulnar sesamoids. Owls, Vultures and Hawks do not seem generally to possess them. The ulnar joint of the American Sea Eagle (that bird as Franklin called it "of bad moral character," which for an inexplicable reason the United States has chosen for its emblem) has an ulnar joint as shown in Plate I, Fig. 1 without sesamoid. In the White-tailed Eagle I have, indeed, found a *cartilaginous* sesamoid of the elbow-joint: see Figs. 2 *a* and 2 *b*. In the House Sparrow there is, however, a well-marked ulnar sesamoid: see Figs. 3 *a* and 3 *b*. The drawings seem to suggest that the olecranon is at any rate very much reduced where the ulnar sesamoid exists. I should prefer not to insist on any too sweeping statement till far more elbow-joints in birds have been studied. But it seemed worth while adding these pictures of the elbow-joint and of the ulnar sesamoid as an addendum to the above mentioned paper.

I have to thank very heartily Mr S. Steward of the Royal College of Surgeons for the drawing of the American Sea Eagle, Miss Ida McLearn for those of the White-tailed Eagle, and my colleague, Dr Katherine Watson, for the dissection and drawings of the elbow-joint of the House Sparrow. K. P.

ON AN UNUSUAL CASE OF DIGITAL ANOMALY.

My attention was drawn to this case by Miss L. Eckenstein, the subject being described as a thumbless, but five fingered woman. The subject herself said that she had originally seven toes on each foot, those on the left foot being removed when her right foot was amputated as a child. Dr Julia Bell kindly had a preliminary interview with and accompanied the subject to the radiography, while Mr E. S. Pearson photographed the hands. The following description of the skiagrams and photographs was drawn up by my colleague Dr Percy Stocks. K. P.

The case provides a somewhat unusual combination of congenital anomalies of the extremities. The subject, a female, presented from birth the following abnormalities which are illustrated in the accompanying photographs and skiagrams.

(1) Both hands showed an identical malformation of the thumb, which was represented by a digit with three joints having the characteristics of a finger rather than a thumb. In length and circumference this 1st digit closely resembled the forefinger, from which it was slightly abducted (see Plates I, Fig. 1, IV and V). The metacarpophalangeal joint was almost on a level with that of the neighbouring forefinger, being only about 7 mm. proximal to it instead of about 30 mm. as in a normal thumb. As the photographs indicate, this digit was slightly rotated inwards towards the palm and adductor and opponens muscles were apparently present which enabled the tips of the 1st and 2nd digits to be opposed. The nail was equal in size to that of the 2nd, 3rd and 4th digits.

Examination of the skiagrams (Plate III) shows that, as regards the carpus, the os magnum and trapezium are unusually large and articulate with each other at their proximal ends, separating the small trapezoid from the scaphoid which normally articulate with each other. The proximal end of the 1st metacarpal fits into a V-shaped concavity formed by the distal ends of the trapezium and trapezoid, and articulates equally with both these bones, instead of with the trapezium alone. The proximal end of the 2nd metacarpal fails to articulate with the trapezium, as is usual, but obliquely articulates with the base of the 1st metacarpal. The 1st metacarpal is approximately equal in length to the 3rd, and about 5 mm. shorter than the 2nd. The phalanges are almost exactly similar in size and shape to those of the index finger, but are somewhat curved with concavity towards the central axis of the hand.

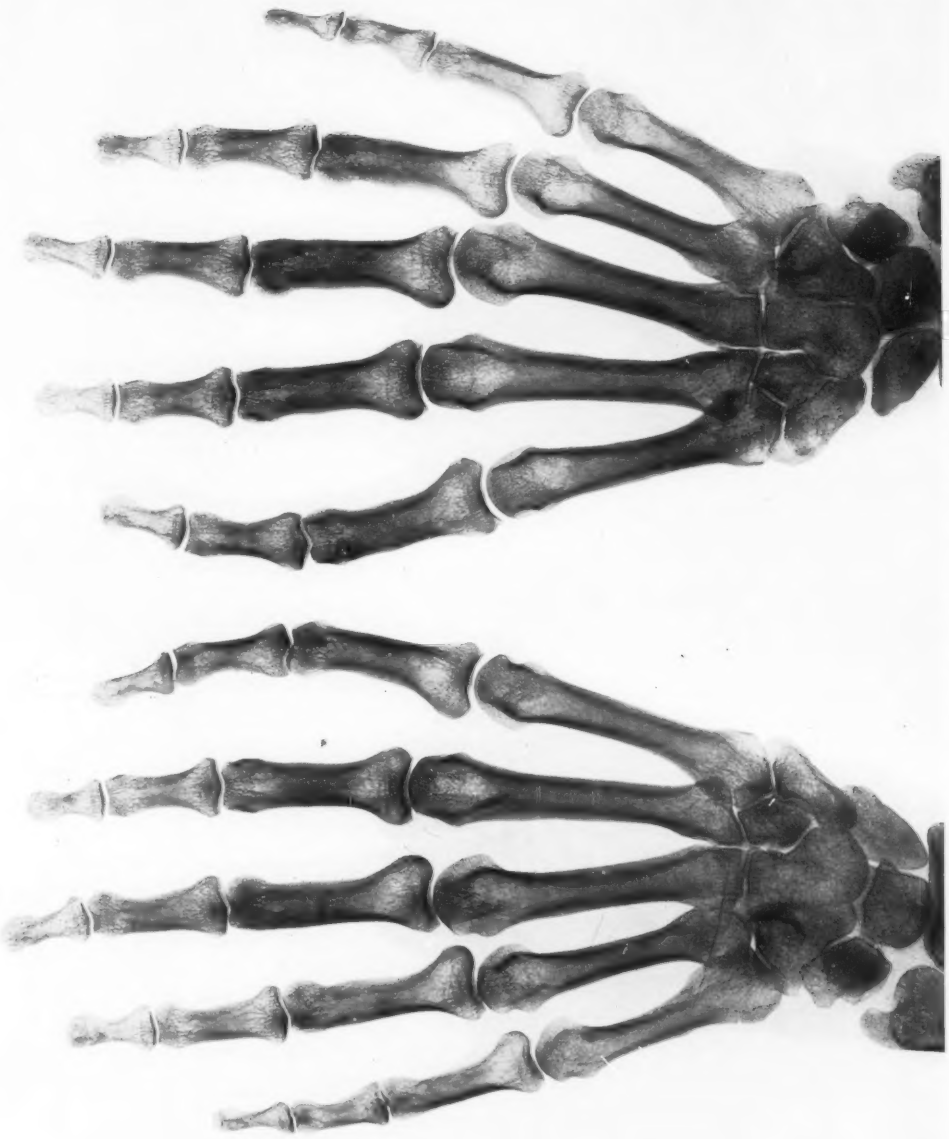
(2) A photograph (Plate I, Fig. 2) taken when the patient was a girl shows that there was apparently congenital absence of the right tibia, whilst the fibula, presumably present, was only about a quarter of its normal length. This was accompanied by extreme equino-varus of the right foot, as is usual in such cases. The great toe of this foot was abnormally broad, and has the appearance of being grooved longitudinally, whilst on its pre-axial side is a complete supernumerary digit with an appendage of skin along its inner border which probably represents a rudimentary 7th digit. This deformed limb was amputated at the knee in youth, and further details are not available.

Fig. 1.

Fig. 2.

Subject as a child, before amputation of right foot or supernumerary toes of left. From Photographs by Dr J. Dill Russell, 1897.

Skiagram of Left Foot in its present state.
Radiographed by Dr R. W. A. Salmond, 1922.



Skiagrams of Left and Right Hands. Radiographed by Dr R. W. A. Salmond, 1922.

Fig. 1. Palms of Left and Right Hands. Photographed by E. S. Pearson, 1922. Fig. 2.

Fig. 2.

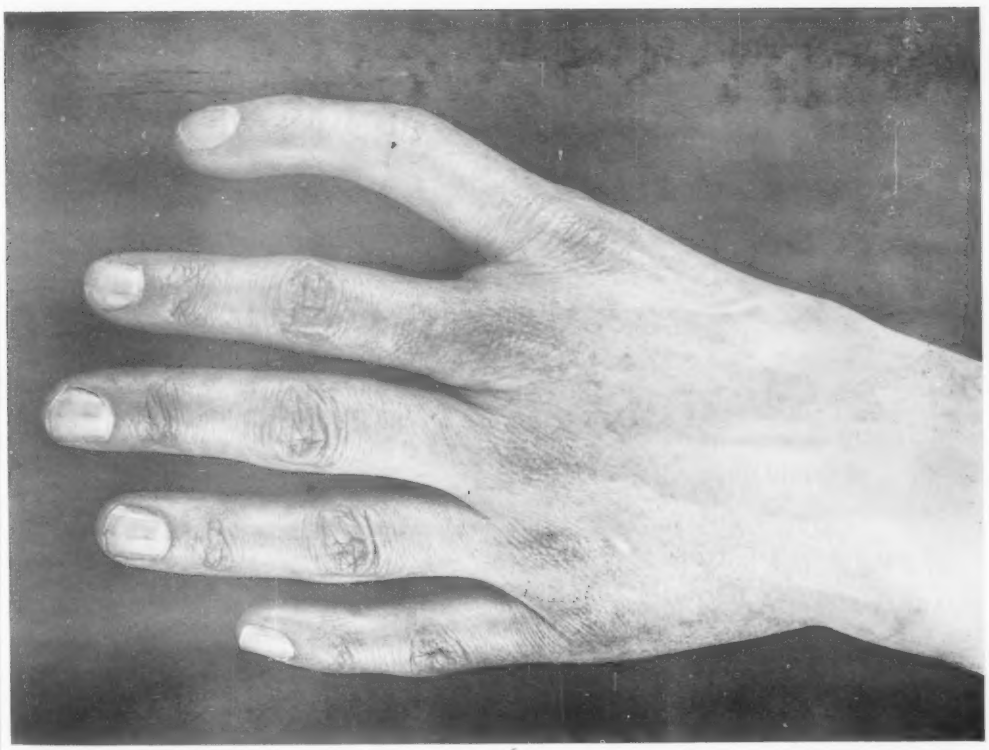


Fig. 1. Backs of Left and Right Hands. Photographed by E. S. Pearson, 1922.

(3) The left foot showed the presence of a rudimentary 6th digit on the pre-axial side of the great toe, which was evidenced by a bony projection on the inner side of the last phalanx, and another near the base of the metatarsal. There was also a skin appendage on the phalangeal projection which possibly represents the rudiments of a 7th digit. A skiagram (Plate II) of this foot shows that the internal cuneiform bone is enlarged on its inner side, having an additional articular surface for another metatarsal bone on inner side of the metatarsal of great toe. There is also an appearance of a longitudinal groove separating these portions of the bone. The metatarsal and 1st phalanx of the great toe are normal in outline, and two sesamoids are present near the proximal end of the former. Articulating with the internal cuneiform and base of 1st metatarsal is an additional metatarsal bone, ending after about 25 mm. in an attenuated stump. Attached by a bridge of bone to the inner side of the last phalanx of the great toe is an additional phalanx, which articulates with the 1st phalanx of the great toe. There appear to be one or more sesamoids near the base of this extra phalanx.

All the anomalies in this case were pre-axial, and with exception of the absence of tibia, symmetrical. Deficiency of the tibia is frequently associated with defective development of the great toe; its association with supernumerary toes on this side of the foot is exceptional. As regards the hand and foot abnormalities, two cases have been reported which resemble this case to some extent, both of which had hereditary history of polydactylism. McKellar(1) records the case of a female (VI. 4 in Lewis's pedigree of this family(3)), in whom the 1st digit of each hand had a metacarpal and three phalanges, whilst in each foot the great toe was bifid, and webbed from the metatarsal onwards. Struthers(2) mentions a case in one of his families (II. 3 in Lewis's pedigree(3)) who had six toes on each foot and "very long thumbs."

In the present case the subject states that to her knowledge there is no history of any hereditary anomaly of hand or foot. It is worth noting that she finds no difficulty in carrying on her profession of dressmaker, she opposes forefinger and "finger thumb" as easily and successfully as most persons do the forefinger and thumb.

We have to thank the subject for the fulness of the information she has given us with regard to her relatives, and their freedom from anomaly. We are deeply indebted to Dr R. W. A. Salmond for his skiagrams of hands and foot; and to Dr J. Dill Russell for photographs he took of the subject as a child before amputation of the right foot, and removal of the supernumerary toes on the side of the big toe of the left foot by the late Professor Christopher Heath. No publication with regard to the matter appears then to have been made.

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MISCELLANEA.

I. On the Correction necessary for the Correlation Ratio η*.

By KARL PEARSON, F.R.S.

(1) In a paper published in *Biometrika*, Vol. VIII. p. 254, I have dealt with the mean value of η, the correlation ratio in samples from material in which the actual value of η is zero, and shewn that if κ be the number of arrays, then when we sample uncorrelated material the mean value of η is $(\kappa-1)/N$, where N is the size of the sample. That paper, however, did not deal more than superficially, and incorrectly, with the case of the mean value of η when η is not zero in the sampled population. I propose to discuss this problem at greater length in the present paper, noting that the chief difficulty lies in being quite certain that all the terms necessary to our degree of approximation have been really included. The general rule I think is that in proceeding to terms of the order $1/N$ we must keep not only first powers of statistical differentials but squares and products of statistical differentials; while if we wish to go to the next higher approximation we must retain not only cubes but fourth powers of statistical differentials. Our purpose at present is to deal only with large samples, and it will be adequate to proceed only to squares and products of such differentials.

(2) Let there be two variables x, y, means $\bar{x}$, $\bar{y}$, standard-deviations σ_x, σ_y, correlation coefficient r; we will take x_s for any particular value of x, and y_t for a particular value of y; the corresponding marginal frequencies being n_{x_s}, n_{y_t}, and the mean and standard-deviation of the array of y's corresponding to x_s being $\bar{y}_{x_s}$ and $\sigma_{y_{x_s}}$. We shall use η for the correlation ratio $\eta_{y,x}$ of y on x. Then by definition:

$$\eta^2=\frac{S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}}{N\sigma_y{}^2} \quad \text{.....(i),}$$

where for brevity we write

$$\sigma_M{}^2=S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}/N \quad \text{.....(ii),}$$

while

$$\sigma_y{}^2=S\{n_{y_t}(y_t-\bar{y})^2\}/N \quad \text{.....(iii).}$$

All statistical differentials shall denote the excess the given quantity has in a sample over its value for the sampled population, and when a short quantity is placed above a symbol, that symbol will be supposed to refer to the sampled population. Starting from $\eta^2=\sigma_M{}^2/\sigma_y{}^2$, we have

$$\breve{\eta}^2+d\eta^2=\frac{\breve{\sigma}_M{}^2+\delta\sigma_M{}^2}{\breve{\sigma}_y{}^2+\delta\sigma_y{}^2}=\frac{\breve{\sigma}_y{}^2\delta\sigma_M{}^2-\breve{\sigma}_M{}^2\delta\sigma_y{}^2}{\breve{\sigma}_y{}^2(\breve{\sigma}_y{}^2+\delta\sigma_y{}^2)}+\breve{\eta}^2.$$

Thus:

$$d\eta^2=\left(\frac{\delta\sigma_M{}^2}{\breve{\sigma}_y{}^2}-\breve{\eta}^2\frac{\delta\sigma_y{}^2}{\breve{\sigma}_y{}^2}\right)\left(1-\frac{\delta\sigma_y{}^2}{\breve{\sigma}_y{}^2}+\frac{(\delta\sigma_y{}^2)^2}{\breve{\sigma}_y{}^4}\right)$$

$$=\frac{\delta\sigma_M{}^2}{\breve{\sigma}_y{}^2}-\breve{\eta}^2\frac{\delta\sigma_y{}^2}{\breve{\sigma}_y{}^2}+\breve{\eta}^2\frac{(\delta\sigma_y{}^2)^2}{\breve{\sigma}_y{}^4}-\frac{\delta\sigma_M{}^2\delta\sigma_y{}^2}{\breve{\sigma}_y{}^4}$$

$$+\text{third order terms} \quad \text{.....(iv).}$$

Denoting mean values by square brackets we have to our order of approximation:

$$[\delta\eta^2]=\frac{[\delta\sigma_M{}^2]}{\breve{\sigma}_y{}^2}-\breve{\eta}^2\frac{[\delta\sigma_y{}^2]}{\breve{\sigma}_y{}^2}+\breve{\eta}^2\frac{[(\delta\sigma_y{}^2)^2]}{\breve{\sigma}_y{}^4}-\frac{[\delta\sigma_M{}^2\delta\sigma_y{}^2]}{\breve{\sigma}_y{}^4} \quad \text{.....(iv)}^{\text{bis}}.$$

We have accordingly to find four mean quantities, $[\delta\sigma_M{}^2]$, $[\delta\sigma_y{}^2]$, $[(\delta\sigma_y{}^2)^2]$ and $[\delta\sigma_M{}^2\delta\sigma_y{}^2]$. Consider first

$$\bar{y}_{x_s}=\frac{S(n_{x_s y_t}y_t)}{\breve{n}_{x_s}}.$$

* Reproduced from lecture notes.

Accordingly $$\delta\tilde{y}_{x_s}=\frac{S\{\delta n_{x_s\nu t}y_t\}}{\check{n}_{x_s}}\left(1-\frac{\delta n_{x_s}}{\check{n}_{x_s}}\right)-\tilde{y}_{x_s}\frac{\delta n_{x_s}}{\check{n}_{x_s}}\left(1-\frac{\delta n_{x_s}}{\check{n}_{x_s}}\right) \quad \text{.....(v),}$$

to our order of approximation.

From this it follows, remembering that*

$$[\delta n_{x_s\nu t}]=0, \qquad [\delta n_{x_s\nu t}\,\delta n_{x_s}]=\check{n}_{x_s\nu t}\left(1-\frac{\check{n}_{x_s}}{N}\right), \qquad [(\delta n_{x_s})^2]=\check{n}_{x_s}\left(1-\frac{\check{n}_{x_s}}{N}\right),$$

that $$[\delta\tilde{y}_{x_s}]=0 \quad \text{.....(vi).}$$

Again from (v):

$$\delta\tilde{y}_{x_s}=\frac{S(\delta n_{x_s\nu t}y_t)}{\check{n}_{x_s}}-\frac{\tilde{y}_{x_s}}{\check{n}_{x_s}}\delta n_{x_s}+\text{square and higher terms}$$

$$=\frac{S\{(\delta n_{x_s\nu t})(y_t-\tilde{y}_{x_s})\}}{\check{n}_{x_s}}+\ldots,$$

$$\delta\tilde{y}_{x_s}{}^2=\frac{S\{(\delta n_{x_s\nu t})^2(y_t-\tilde{y}_{x_s})^2\}}{\check{n}_{x_s}{}^2}+\frac{2S'\{(\delta n_{x_s\nu t}\,\delta n_{x_s\nu t'})(y_t-\tilde{y}_{x_s})(y_{t'}-\tilde{y}_{x_s})\}}{\check{n}_{x_s}{}^2}+\text{third order terms,}$$

where S' is a summation for all different values of t and t'.

Or: $$[(\delta\tilde{y}_{x_s})^2]=\frac{S\left\{\check{n}_{x_s\nu t}\left(1-\frac{\check{n}_{x_s\nu t}}{N}\right)(y_t-\tilde{y}_{x_s})^2\right\}}{\check{n}_{x_s}{}^2}-\frac{2S'\{\check{n}_{x_s\nu t}\check{n}_{x_s\nu t'}(y_t-\tilde{y}_{x_s})(y_{t'}-\tilde{y}_{x_s})\}}{N\check{n}_{x_s}{}^2}+\text{etc.},$$

remembering $$[\delta n_{x_s\nu t}\,\delta n_{x_s\nu t'}]=-\frac{\check{n}_{x_s\nu t}\check{n}_{x_s\nu t'}}{N},$$

$$=\frac{\check{\sigma}_{\nu x_s}{}^2}{\check{n}_{x_s}}-\left\{\frac{S\{\check{n}_{x_s\nu t}(y_t-\tilde{y}_{x_s})\}}{\check{n}_{x_s}}\right\}^2\frac{1}{N}+\text{etc.}$$

But $$S\{n_{x_s\nu t}(y_t-\tilde{y}_{x_s})\}=0.$$

Hence $$[(\delta\tilde{y}_{x_s})^2]=\check{\sigma}_{\nu x_s}{}^2/\check{n}_{x_s} \quad \text{.....(vii),}$$

i.e. the same value as if n_{x_s} were constant in all the samples.

Again $[(\delta\bar{y})^2]$ is the standard-deviation squared of means of samples $=\check{\sigma}_y{}^2/N$. Further we shall require $[\delta\tilde{y}_{x_s}\,\delta\bar{y}]$. Now:

$$\delta\tilde{y}_{x_s}=\frac{S\{\delta n_{x_s\nu t}(y_t-\tilde{y}_{x_s})\}}{\check{n}_{x_s}}+\text{etc.},$$

$$\delta\tilde{y}_{x_{s'}}=\frac{S\{\delta n_{x_{s'}\nu t}(y_t-\tilde{y}_{x_{s'}})\}}{\check{n}_{x_{s'}}}+\text{etc.},$$

$$[\delta\tilde{y}_{x_s}\,\delta\tilde{y}_{x_{s'}}]=-\frac{SS\{\check{n}_{x_s\nu t}\check{n}_{x_{s'}\nu t'}(y_t-\tilde{y}_{x_s})(y_{t'}-\tilde{y}_{x_{s'}})\}}{N\check{n}_{x_s}\check{n}_{x_{s'}}}+\text{third order terms}$$

$$=-\frac{S\{\check{n}_{x_s\nu t}(y_t-\tilde{y}_{x_s})\}}{N\check{n}_{x_s}}\times\frac{S\{\check{n}_{x_{s'}\nu t'}(y_{t'}-\tilde{y}_{x_{s'}})\}}{\check{n}_{x_{s'}}N}+\text{etc.}$$

Or, $[\delta\tilde{y}_{x_s}\,\delta\tilde{y}_{x_{s'}}]=0$, because both these factors vanish .....(viii).

Now consider $$[\delta\tilde{y}_{x_s}\,\delta n_{x_s}]=\frac{S\{[\delta n_{x_s\nu t}\,\delta n_{x_s}](y_t-\tilde{y}_{x_s})\}}{\check{n}_{x_s}}$$

$$=S\left\{\frac{\check{n}_{x_s\nu t}}{\check{n}_{x_s}}\left(1-\frac{\check{n}_{x_s}}{N}\right)(y_t-\tilde{y}_{x_s})\right\}$$

$$=\frac{1}{\check{n}_{x_s}}\left(1-\frac{\check{n}_{x_s}}{N}\right)S\{\check{n}_{x_s\nu t}(y_t-\tilde{y}_{x_s})\}$$

$$=0 \quad \text{.....(ix).}$$

* *Biometrika*, Vol. III. p. 273 *et seq.*

Similarly
$$[\delta\bar{y}_{x_s}\delta n_{x_{s'}}]=\frac{S\{[\delta n_{x_s y_t}\delta n_{x_{s'}}](y_t-\bar{y}_{x_s})\}}{n_{x_s}}$$
$$=-\frac{n_{x_{s'}}}{N}\frac{S\{n_{x_s y_t}(y_t-\bar{y}_{x_s})\}}{n_{x_s}}$$
$$=0 \qquad \text{(x)*}$$

Now
$$\delta\bar{y}=\frac{S(\delta n_{x_s}\bar{y}_{x_s})+S(n_{x_s}\delta\bar{y}_{x_s})}{N}.$$

Multiply by $\delta\bar{y}_{x_s}$ and take the mean, then by (viii), (ix) and (x) above:
$$[\delta\bar{y}_{x_s}\delta\bar{y}]=\frac{n_{x_s}}{N}[(\delta\bar{y}_{x_s})^2]=\frac{\sigma_{y_{x_s}}^2}{N} \qquad \text{(xi)}.$$

Combining these results we have
$$[(\delta\bar{y}_{x_s}-\delta\bar{y})^2]=\frac{\sigma_{y_{x_s}}^2}{n_{x_s}}-\frac{2}{N}\sigma_{y_{x_s}}^2+\frac{\sigma_y^2}{N} \qquad \text{(xii)}.$$

We can now obtain the mean value of σ_M^2 in samples. Since
$$\sigma_M^2=S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}/N,$$
$$\sigma_M^2+\delta\sigma_M^2=S\{(n_{x_s}+\delta n_{x_s})(\bar{y}_{x_s}-\bar{y}+\delta\bar{y}_{x_s}-\delta\bar{y})^2\}/N,$$
and
$$N\delta\sigma_M^2=S\{\delta n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}+2S\{(\bar{y}_{x_s}-\bar{y})n_{x_s}(\delta\bar{y}_{x_s}-\delta\bar{y})\}$$
$$+2S\{(\bar{y}_{x_s}-\bar{y})\delta n_{x_s}(\delta\bar{y}_{x_s}-\delta\bar{y})\}$$
$$+S\{n_{x_s}(\delta\bar{y}_{x_s}-\delta\bar{y})^2\}$$
$$+\text{terms of third order} \qquad \text{(xiii)}.$$

The summations of the first line both vanish for they contain linear terms in δn_{x_s}, $\delta\bar{y}_{x_s}$ and $\delta\bar{y}$ which vanish when we take mean values. We have shewn that $[\delta n_{x_s}\delta\bar{y}_{x_s}]$ is zero. The mean value of $[\delta n_{x_s}\delta\bar{y}]$ is to be found from
$$[\delta n_{x_s}\delta\bar{y}]=[\delta n_{x_s}S(\delta n_{y_t}y_t)]/N$$
$$=S_t\left(n_{x_s y_t}-\frac{n_{x_s}n_{y_t}}{N}\right)y_t/N$$
$$=\frac{n_{x_s}}{N}(\bar{y}_{x_s}-\bar{y}).$$

Accordingly the mean of the second line of the expression in (xiii) becomes
$$-2S\{n_{x_s}(\bar{y}_{x_s}-\bar{y})^2\}/N=-2\sigma_M^2=-2\eta^2\sigma_y^2,$$
and we have
$$N[\delta\sigma_M^2]=-2\eta^2\sigma_y^2+S\left\{n_{x_s}\left(\frac{\sigma_{y_{x_s}}^2}{n_{x_s}}-\frac{2}{N}\sigma_{y_{x_s}}^2+\frac{\sigma_y^2}{N}\right)\right\}.$$

Now $S(n_{x_s})=N$, $S(n_{x_s}\sigma_{y_{x_s}}^2)/N=$weighted mean square standard-deviation of the arrays $=(1-\eta^2)\sigma_y^2$; and, if there be κ arrays, $S(\sigma_{y_{x_s}}^2)/\kappa=$unweighted mean square standard-deviation of arrays$=\lambda_1(1-\eta^2)\sigma_y^2$, let us say, where λ_1 will not differ much from unity and will equal unity if $\bar{\eta}$ be zero, or η be zero, or if there be homoscedasticity. Accordingly
$$\frac{[\delta\sigma_M^2]}{\sigma_y^2}=\frac{(\kappa\lambda_1-2)(1-\eta^2)+1-2\eta^2}{N} \qquad \text{(xiv)}.$$

We now take the second term of (iv)$^{\text{bis}}$ and find very easily†:
$$\frac{[\delta\sigma_y^2]}{\sigma_y^2}=-\frac{1}{N} \qquad \text{(xv)}.$$

* See "Theory of Skew Correlation," *Drapers' Company Research Memoirs, Biometric Series*, No. XIV, p. 13, where (viii)—(x) have been previously demonstrated.

† This is really the familiar relation of text-books of theory of observations that mean σ_y^2 of samples $=\sigma_y^2(N-1)/N$.

Hence $$\frac{[\delta\sigma_M{}^2]}{\breve{\sigma}_y{}^2}-\breve{\eta}^2\frac{[\delta\sigma_y{}^2]}{\breve{\sigma}_y{}^2}=\frac{(\kappa\lambda_1-1)(1-\breve{\eta}^2)}{N} \quad \text{.....(xvi)*.}$$

The third term of (iv)^{bis} involves $[(\delta\sigma_y{}^2)^2]$, or the standard-deviation of the second moment coefficient μ_2 of the y-variable. But this equals $(\mu_4-\mu_2{}^2)/N=\breve{\sigma}_y{}^4(\beta_2-1)/N$. Accordingly the third term is

$$\breve{\eta}^2[(\delta\sigma_y{}^2)^2]/\breve{\sigma}_y{}^4=\breve{\eta}^2(\beta_2-1)/N \quad \text{.....(xvii).}$$

It now remains to find the last term depending on $[\delta\sigma_M{}^2\,\delta\sigma_y{}^2]$.

Remembering that $\sigma_y{}^2=S\{n_{y_t}(y_t-\bar{y})^2\}/N$, we easily find

$$\delta\sigma_y{}^2=\frac{S\{\delta n_{y_t}(y_t-\bar{y})^2\}}{N}-\frac{2\delta\bar{y}\,.\,S\{\breve{n}_{y_t}(y_t-\bar{y})\}}{N}+\text{terms of second order}$$

$$=\frac{S\{\delta n_{y_t}(y_t-\bar{y})^2\}}{N}+\text{terms of second order} \quad \text{.....(xviii).}$$

Similarly from (ii):

$$\delta\sigma_M{}^2=\frac{S\{\delta n_{x_s}(\tilde{y}_{x_s}-\bar{y})^2\}}{N}+\frac{2S\{\breve{n}_{x_s}(\tilde{y}_{x_s}-\bar{y})\,\delta\tilde{y}_{x_s}\}}{N}-\frac{2\delta\bar{y}\,S\{\breve{n}_{x_s}(\tilde{y}_{x_s}-\bar{y})\}}{N}+\text{terms of second order}$$

$$=\frac{S\{\delta n_{x_s}(\tilde{y}_{x_s}-\bar{y})^2\}}{N}+\frac{2S\{\breve{n}_{x_s}(\tilde{y}_{x_s}-\bar{y})\,\delta\tilde{y}_{x_s}\}}{N}+\text{terms of second order} \quad \text{.....(xix).}$$

When we take the product it will only be needful to take terms of the first order in $\delta\sigma_y{}^2$ and $\delta\sigma_M{}^2$, if we neglect third order terms.

The sum in (xviii) refers to t and in (xix) to s.

We consider in the first place the mean $[\delta n_{x_s}\,\delta n_{y_t}]$. But this equals

$$\breve{n}_{x_s y_t}-\frac{\breve{n}_{x_s}\breve{n}_{y_t}}{N} \quad \text{.....(xx).}$$

We next deal with $[\delta n_{y_t}\,\delta\tilde{y}_{x_s}]$. But

$$\delta\tilde{y}_{x_s}=S\{\delta n_{x_s y_{t'}}(y_{t'}-\tilde{y}_{x_s})\}/\breve{n}_{x_s}+\text{etc.},$$

$$\therefore\ [\delta n_{y_t}\,\delta\tilde{y}_{x_s}]=S\{[\delta n_{x_s y_{t'}}\,\delta n_{y_t}](y_{t'}-\tilde{y}_{x_s})\}/\breve{n}_{x_s}$$

$$=\breve{n}_{x_s y_t}\left(1-\frac{\breve{n}_{y_t}}{N}\right)\left(\frac{y_t-\tilde{y}_{x_s}}{\breve{n}_{x_s}}\right)-\frac{S\{\breve{n}_{x_s y_{t'}}\breve{n}_{y_t}\}}{N}\frac{(y_{t'}-\tilde{y}_{x_s})}{\breve{n}_{x_s}}$$

$$=\frac{\breve{n}_{x_s y_t}(y_t-\tilde{y}_{x_s})}{\breve{n}_{x_s}}-\frac{\breve{n}_{y_t}}{N}\frac{S\{\breve{n}_{x_s y_{t'}}(y_{t'}-\tilde{y}_{x_s})\}}{\breve{n}_{x_s}}+\text{terms of third order.}$$

Here the second summation is clearly zero, or to our approximation

$$[\delta n_{y_t}\,\delta\tilde{y}_{x_s}]=\frac{\breve{n}_{x_s y_t}(y_t-\tilde{y}_{x_s})}{\breve{n}_{x_s}} \quad \text{.....(xxi).}$$

Using (xx) we have:

$$[\delta n_{x_s}\delta\sigma_y{}^2]=S_t\left\{\left(\breve{n}_{x_s y_t}-\frac{\breve{n}_{x_s}\breve{n}_{y_t}}{N}\right)\frac{(y_t-\bar{y})^2}{N}\right\}$$

$$=\breve{n}_{x_s}\{\sigma_{y_{x_s}}{}^2+(\tilde{y}_{x_s}-\bar{y})^2-\sigma_y{}^2\}/N \quad \text{.....(xxii),}$$

and from (xxi)

$$[\delta\tilde{y}_{x_s}\delta\sigma_y{}^2]=S_t\{\breve{n}_{x_s y_t}(y_t-\tilde{y}_{x_s})(y_t-\bar{y})^2\}/(N\breve{n}_{x_s})$$

$$=S_t(\breve{n}_{x_s y_t}\{(y_t-\tilde{y}_{x_s})^3+2(\tilde{y}_{x_s}-\bar{y})(y_t-\tilde{y}_{x_s})^2$$

$$+(\tilde{y}_{x_s}-\bar{y})^2(y_t-\tilde{y}_{x_s})\})/(\breve{n}_{x_s}N).$$

Now if $(\breve{\mu}_3)_{y_{x_s}}$ be the third moment coefficient about its mean of the x_s array of y's for the sampled population, and if we note that $S_t\{\breve{n}_{x_s y_t}(y_t-\tilde{y}_{x_s})\}=0$, we can write this result:

$$[\delta\tilde{y}_{x_s}\delta\sigma_y{}^2]=\{(\breve{\mu}_3)_{y_{x_s}}+2(\tilde{y}_{x_s}-\bar{y})\breve{\sigma}_{y_{x_s}}{}^2\}/N \quad \text{.....(xxiii).}$$

* This agrees with the value given in *Biometrika*, *loc. cit.*, where the squares of the statistical differentials were neglected.

We have now to multiply (xix) by $\delta\sigma_y^2$, take the mean and use (xxii) and (xxiii). We find:

$$[\delta\sigma_M^2\,\delta\sigma_y^2]=\frac{1}{N^2}S_s\{\breve{n}_{x_s}\breve{\sigma}_{y_{x_s}}^2(\breve{y}_{x_s}-\bar{y})^2+\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^4-\breve{\sigma}_y^2\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^2$$
$$+2\breve{n}_{x_s}(\breve{\mu}_3)_{y_{x_s}}(\breve{y}_{x_s}-\bar{y})+4\breve{n}_{x_s}\breve{\sigma}_{y_{x_s}}^2(\breve{y}_{x_s}-\bar{y})^2\}\text{.....(xxiv).}$$

It remains to consider these terms individually. First:

$$S\{\breve{n}_{x_s}\breve{\sigma}_{y_{x_s}}^2(\breve{y}_{x_s}-\bar{y})^2\}.$$

If the system were homoscedastic this would equal

$$(1-\breve{\eta}^2)\breve{\sigma}_y^2\times S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^2\}=(1-\breve{\eta}^2)\breve{\sigma}_y^2\sigma_M^2N=\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4N.$$

Accordingly let us take

$$\lambda_2=\frac{S\{\breve{n}_{x_s}\breve{\sigma}_{y_{x_s}}^2(\breve{y}_{x_s}-\bar{y})^2\}}{N\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4},$$

and remember that λ_2 will not as a rule differ widely from unity.

Secondly: $$S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^4\}.$$

If the system were linear this would equal

$$\frac{S\{\breve{n}_{x_s}\breve{r}^4(x-\bar{x})^4\breve{\sigma}_y^4\}}{\breve{\sigma}_x^4}=\frac{\breve{r}^4\breve{\sigma}_y^4}{\breve{\sigma}_x^4}N(\mu_4)_x=\breve{r}^4N\beta_2\breve{\sigma}_y^4,$$

where β_2 is the β_2 coefficient (i.e. μ_4/μ_2^2) of the x-variable and will not be very widely different from 3.

Accordingly let us take

$$\lambda_3=\frac{S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^4\}}{3\breve{\eta}^4N\breve{\sigma}_y^4},$$

and remember that λ_3 may be put approximately unity.

Thirdly: $$S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^2\}=N\breve{\sigma}_M^2=N\breve{\eta}^2\breve{\sigma}_y^2.$$

Lastly: $$S\{\breve{n}_{x_s}(\breve{\mu}_3)_{y_{x_s}}(\breve{y}_{x_s}-\bar{y})\}.$$

Now $(\breve{\mu}_3)_{y_{x_s}}$ would be zero, if the arrays were symmetrical, and will generally be small. If the arrays have equal skewness the expression equals

$$(\breve{\mu}_3)_{y_{x_s}}S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})\}=0.$$

Generally we may take it equal to $\lambda_4\sigma_y^4N$, where λ_4 will usually be very small.

Thus finally we may write:

$$\frac{[\delta\sigma_M^2\,\delta\sigma_y^2]}{\breve{\sigma}_y^4}=\frac{1}{N}\{5\lambda_2\breve{\eta}^2(1-\breve{\eta}^2)+3\lambda_3\breve{\eta}^4-\breve{\eta}^2+2\lambda_4\}\text{ .....(xxv).}$$

We can now marshal our results to obtain $[\delta\eta^2]$ from (xvi), (xvii) and (xxv). We have:

$$[\delta\eta^2]=\frac{1}{N}\{\kappa\lambda_1-1)(1-\breve{\eta}^2)+\breve{\eta}^2(\beta_2-1)-5\lambda_2\breve{\eta}^2(1-\breve{\eta}^2)-3\lambda_3\breve{\eta}^4+\breve{\eta}^2-2\lambda_4\}$$
$$=\frac{1}{N}\{(\kappa\lambda_1-1)(1-\breve{\eta}^2)+\breve{\eta}^2(\beta_2-5\lambda_2)+\breve{\eta}^4(5\lambda_2-3\lambda_3)-2\lambda_4\}\text{ .....(xxvi),}$$

where, collecting out values:

κ = number of arrays,

$\lambda_1=\dfrac{S(\breve{\sigma}_{y_{x_s}}^2)/\kappa}{(1-\breve{\eta}^2)\breve{\sigma}_y^2}$ = unity approximately,

$\beta_2=\beta_2$-coefficient of y-variable = 3 approximately,

$\lambda_2=\dfrac{S\{\breve{n}_{x_s}\breve{\sigma}_{y_{x_s}}^2(\breve{y}_{x_s}-\bar{y})^2}{N\breve{\eta}^2(1-\breve{\eta}^2)\breve{\sigma}_y^4}$ = unity approximately,

$\lambda_3=\dfrac{S\{\breve{n}_{x_s}(\breve{y}_{x_s}-\bar{y})^4\}}{3\breve{\eta}^4N\breve{\sigma}_y^4}$ = unity approximately,

$\lambda_4=\dfrac{S\{\breve{n}_{x_s}(\breve{\mu}_3)_{y_{x_s}}(\breve{y}_{x_s}-\bar{y})\}}{N\breve{\sigma}_y^4}$ = zero approximately.

Now it is clear that the actual values of these constants could be calculated, if with considerable labour, for the given sample and their values used instead of those of the sampled population as approximations, even as we calculate such expressions in probable errors, when the actual state of the sampled population is unknown. But for many purposes it will be quite adequate, since we are supposing the sample a large one, to insert the approximate values given on the right-hand side above, as they will be divided by N. In that case we have

$$[\delta\eta^2]=\frac{1-\bar{\eta}^2}{N}(\kappa-1-2\bar{\eta}^2) \quad \text{.....(xxvii).}$$

This result differs from my previous one, already referred to, by the presence of the $2\bar{\eta}^2$.

If $\hat{\eta}^2$ be the mean η^2 of samples, we have

$$1-\hat{\eta}^2=1-(\bar{\eta}^2+\delta\eta^2)=(1-\bar{\eta}^2)\left\{1-\frac{\kappa-3+2(1-\bar{\eta}^2)}{N}\right\}.$$

Or, if $\quad \hat{\epsilon}=1-\hat{\eta}^2$, and $\bar{\epsilon}=1-\bar{\eta}^2$,

$$\hat{\epsilon}=\bar{\epsilon}\left(1-\frac{\kappa-3}{N}\right)-\frac{2}{N}\bar{\epsilon}^2 \quad \text{.....(xxviii),}$$

which is a quadratic to determine $\bar{\epsilon}$ in terms of $\hat{\epsilon}$; the required root since $\hat{\epsilon}$ and $\bar{\epsilon}$ must vanish simultaneously is

$$\bar{\epsilon}=\frac{-\sqrt{\{1-(\kappa-3)/N\}^2-(8/N)\hat{\epsilon}}+\{1-(\kappa-3/N)\}}{4/N}.$$

Or, approximately:

$$\bar{\eta}^2=\frac{\hat{\eta}^2-(\kappa-3)/N}{1-(\kappa-3)/N}-\frac{2(1-\hat{\eta}^2)^2}{N\{1-(\kappa-3)/N\}^3} \quad \text{.....(xxix).}$$

Now equation (xxix) is a relation between the mean $\hat{\eta}^2$ for all samples and the $\bar{\eta}^2$ of the sampled population. In a practical example neither of these quantities is really known. We want to find $\bar{\eta}^2$, and all we are given is the value of η^2 in a single particular sample. It might be more or less reasonable to assume that the η^2 observed in the sample was the *modal* value of the η^2 for all samples. But so far there has been no discussion of the frequency curve for the distribution of η^2 in samples, and such discussion cannot be undertaken until we have greater knowledge of the form of skew bivariate frequency distributions. We do know that in the case of normal correlation the modal value of r^2 in samples differs from the value of r^2 in the sampled population by terms of the order $1/N$ and higher inverse powers. It may be taken for granted that modal η^2 in samples will differ from mean η^2 in samples—i.e. our $\hat{\eta}^2$—by a term of order $1/N$. Accordingly if we replace $\hat{\eta}^2$ by modal η^2 and eventually by the value in the single sample, we should not be justified in keeping the second or $(1-\bar{\eta}^2)^2$ term in (xxix), and we might modify $\kappa-3$ in the first term by something of the order of a unit.

Accordingly it does not seem possible at present to go further than suggest that

$$\frac{\text{Observed } \eta^2-(\kappa-3)/N}{1-(\kappa-3)/N}$$

is a reasonable value for the η^2 of the sampled population, provided N is fairly large. Equation (xxix) would give as it stands a good result, if we could find the mean η^2 for a number of samples. Of course the first consideration in any investigation of η^2 is to determine whether it is comparable with $(\kappa-1)/N$. If it be less than this value we cannot assert significant association. If it be greater than this value we have to consider whether η as observed differs considerably from

$$\sqrt{\frac{\kappa-1}{N}}+\cdot67449\frac{1}{\sqrt{N}},$$

and for general purposes we must settle whether η differs from $\sqrt{(\kappa-1)/N}$ by, say, $1\cdot7/\sqrt{N}$.

II. Further Note on the χ^2 test of Goodness of Fit.

By KARL PEARSON, F.R.S.

I wish to make a correction to my former note on this subject on p. 189 of this volume. Mr Fisher on p. 90 of his paper* wrote:

"In 1915 Greenwood and Yule, using four-fold tables to test the effect of inoculation against typhoid and cholera, followed Pearson in applying Elderton's table with $n'=4$."

In my reply to Mr Fisher, I had not in mind what Greenwood and Yule had done. I only considered that they had applied χ^2 to a four-fold table and it seemed to me that they ought accordingly to have taken $n'=4$, precisely as I personally should do for a four-fold table in general. But on examining Messrs Greenwood and Yule's paper I find that they are not dealing with what I should admit are true four-fold tables at all. *They are questioning whether two sets of alternative categories can be considered as samples of the same population.* In other words they are dealing with a special case of the problem which I have discussed in *Biometrika*, Vol. VIII. p. 250, and they have chosen, I think, somewhat arbitrarily to exhibit their data in a four-fold table form. I consider that their table is not a true four-fold table at all. In such a four-fold table the marginal totals are perfectly free, there are four frequency classes, and if the deviations of these classes from any assumed distribution be taken, then n' will equal 4, and this is the value which must be used to obtain P. This is equally true whether we put our four categories in four-fold order or in serial order.

But clearly as approached from the standpoint of *Biometrika*, Vol. VIII. the problem of Messrs Greenwood and Yule gives $n'=2$. This arises from the fact that they have arbitrarily fixed by the size of their inoculated and uninoculated groups two of the marginal totals. Such marginal totals are not fixed in a true four-fold table, where these marginal totals are subject to the variations of random sampling. On this ground I venture to think it is not wise to speak of such tables as "four-fold tables" when applying the χ^2, P test. This use led the authors to apply χ^2 for $n'=4$ and not $n'=2$. In writing my note in the last number, I supposed the four-fold tables to which Mr Fisher referred as used by Messrs Greenwood and Yule were true four-fold tables with the natural freedom of margins.

It is such tables with which I deal in my Eighteenth Memoir on Evolution. I ask what is the probability that the distribution in four cells—which may be arranged serially or in quadrate form, i.e. with free margins—differs from a given null correlation theoretical distribution, and, as I pointed out in my note—not knowing the marginal distributions of the theoretical sampled population—we do what we always do when discussing probable error, put the unknown constants of the sampled population equal to the observed constants of the sample. This is not a fixing of the marginal totals and n' remains 4, and not the 2 of Messrs Greenwood and Yule's pseudo-four-fold tables.

III. Review: *Frequency Arrays.* By H. E. Soper, M.A.†

In his "Frequency Arrays," Mr Soper, by means of the symbolic expression of certain fundamental ideas, is enabled to demonstrate anew in a few pages many propositions which have been proved from time to time, the exposition of which has involved for the most part lengthy discussion. He hopes that familiarity with his method of treatment will lead to further discoveries in mathematical statistics and allied subjects. His book will appeal more to the mathematical statistician than to the practical statistician, and should appeal to the research worker, because a new world is opened to him where whole sentences are expressed quickly in symbolic forms, and where events may be said to move more rapidly than in the slow-moving

* *Journal of R. Statistical Society*, Vol. LXXXV. p. 90.

† Cambridge University Press.

algebraical world. The book will also appeal to the teacher because it should enable him to demonstrate the truth of many elementary propositions very quickly and easily.

The author starts out by representing the "frequency array" resulting from taking a sample of n individuals (with replacement) from a population containing individuals having characters $A, B, C, \ldots$ in different degrees as $\Sigma p_{abc\ldots} A^a B^b C^c \ldots$, where the general term gives the frequency of occurrence (or the probability of occurrence) of individuals having the character A in degree a, the character B in degree b, the character C in degree c, and so on. The phrase "Frequency Array" he applies not only to the "array" in the ordinary sense of the term, in one variable; but also to describe a Table of Frequency in any number of variables.

When we wish to refer to the point-binomial we either write down a certain number of terms and the general term as follows:

$$p^n,\ np^{n-1}q,\ \frac{n\,.\,n-1}{1\,.\,2}p^{n-2}q^2,\ \ldots\ \frac{n\,!}{r\,!\,n-r\,!}p^{n-r}q^r,\ \ldots$$

or we write down $(p+q)^n$, always understanding that this last expression is supposed expanded, and the connecting positive signs removed, the individual terms representing the frequency of occurrence of 0, 1, 2, ... r, ... successes, in a sample of n, when the chance of a success is q and the chance of a failure is $p=1-q$ for any one "draw" of a sample.

Mr Soper points out that the expression

$$\Sigma \frac{n\,!}{r\,!\,n-r\,!}p^{n-r}q^r A^r = (p+qA)^n$$

represents in itself, without any mental restriction as to the removal of the conjunctive signs, and without any further explanations, the frequency distribution resulting from drawing a sample of n (with replacement) from a population, where the chance of any one draw being a success (A) is q and the chance of its being a failure is p. He points out that the expression $(p+q)^n$ in the ordinary notation is equal to unity and cannot therefore be used and submitted to any algebraic processes; but $(p+qA)^n$ can submit to such processes, with interesting results as the author shews in his book. He also extends this method of presentation to continuous variates and represents for instance the Gaussian frequency distribution of one variate as

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{\sqrt{2\pi}\,.\,\sigma} A^x\,.\,dx,$$

where $\dfrac{e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{\sqrt{2\pi}\,.\,\sigma}dx$ is the frequency of occurrence between x and $x+dx$ of the variate A, and the integral sign replaces the sign of summation in the previous expression. Thus this expression represents the frequency, when the distribution is Gaussian, for all values of x from $-\infty$ to $+\infty$. The author's applications of the ordinary laws of algebra to such expressions give the usual moment equations and so on very easily.

Mr Soper's notation and methods, being perfectly general, enable him easily to obtain the moments of the double hypergeometric series and the usual formulae which occur in problems of multiple regression. He also applies his methods to get the results which have been proved in other problems, such as that of random migration.

There is a great deal of concentrated work in Mr Soper's book, and it is to be hoped that work on the same lines may lead to new discoveries. He has only attempted to illustrate his methods on old problems to shew the uses to which this symbolic notation may be put. The author probably forgets that his readers are not so familiar with his mode of presenting a problem as himself, and contents himself with rather a short preface before plunging into the difficult part of the work. His book might therefore have been improved by a more detailed introduction. Finally we must thank Mr Soper for his short proof of the χ^2 formula, which he, perhaps characteristically, relegates to a footnote. E. C. R.

IV. Review: *Some Formulas in the Theory of Interpolation of Many Independent Variables.* By Professor SEIMATSU NARUMI*.

This paper is mainly of theoretical interest. The author arrives at the extension of Newton's forward and backward interpolation formulae for the case of two variables and also at two-dimensional extensions of the so-called central difference formulae, which are generally attributed to Stirling, Bessel and Gauss, but are in reality due to Newton.

The reader is left to supply much of the algebra and the analysis seems to us somewhat involved, though admirably rigorous.

We refer to the formulae of Bessel, Stirling and Gauss as "so-called" central difference formulae, because we wish to draw a distinction between them and Everett's central difference formula, which, we think, should be regarded as a true central difference formula. Everett's formula seems to possess two important properties:

(1) Only even order differences appear.

(2) Only $2n$ ordinates are needed to obtain correctness up to but not including $(2n+2)$th order differences.

The latter property is shared by the formulae of Bessel and Gauss though Stirling's requires $2n+1$ ordinates for the same accuracy; the former property however only Everett's formula possesses; the other three all containing, implicitly or explicitly, odd order differences.

When we extend Everett's formula to two dimensions property (1) still holds, and we find also that the interpolated value is based on the values of the function at the "nearest" points, a complete set of which always gives accuracy up to, but not including, a given even order of differences.

This property will presumably be shared by the two-dimensional forms of Gauss and Bessel, but not by the extended Stirling formula, and moreover Everett's form possesses the very great advantage (still greater in two dimensions than in one) over all three others, that odd order differences do not occur. This is indeed its distinguishing characteristic.

The extension of Newton's formula is of course not new and this is the only formula in the paper illustrated by a numerical example. Newton's formula is, of course, open to the objection that the interpolated value is not based on the values of the function at the "nearest" points, and we should like to have seen some numerical illustrations of Professor Narumi's other formulae and contrasted them with the two-dimensional form of Everett's central difference formula, given by Professor Pearson. For the above reasons we anticipate that the former will be found less convenient for numerical purposes. J. O. I.

* *Tôhoku Mathematical Journal*, Vol. XVIII. pp. 309–321.

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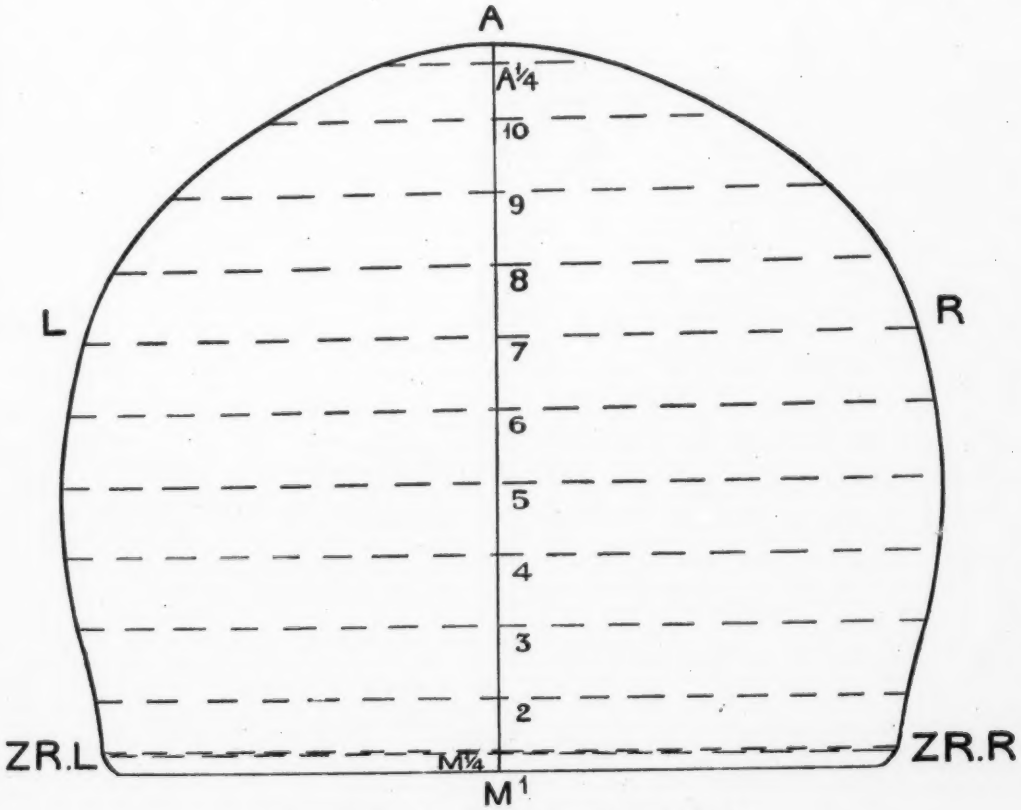

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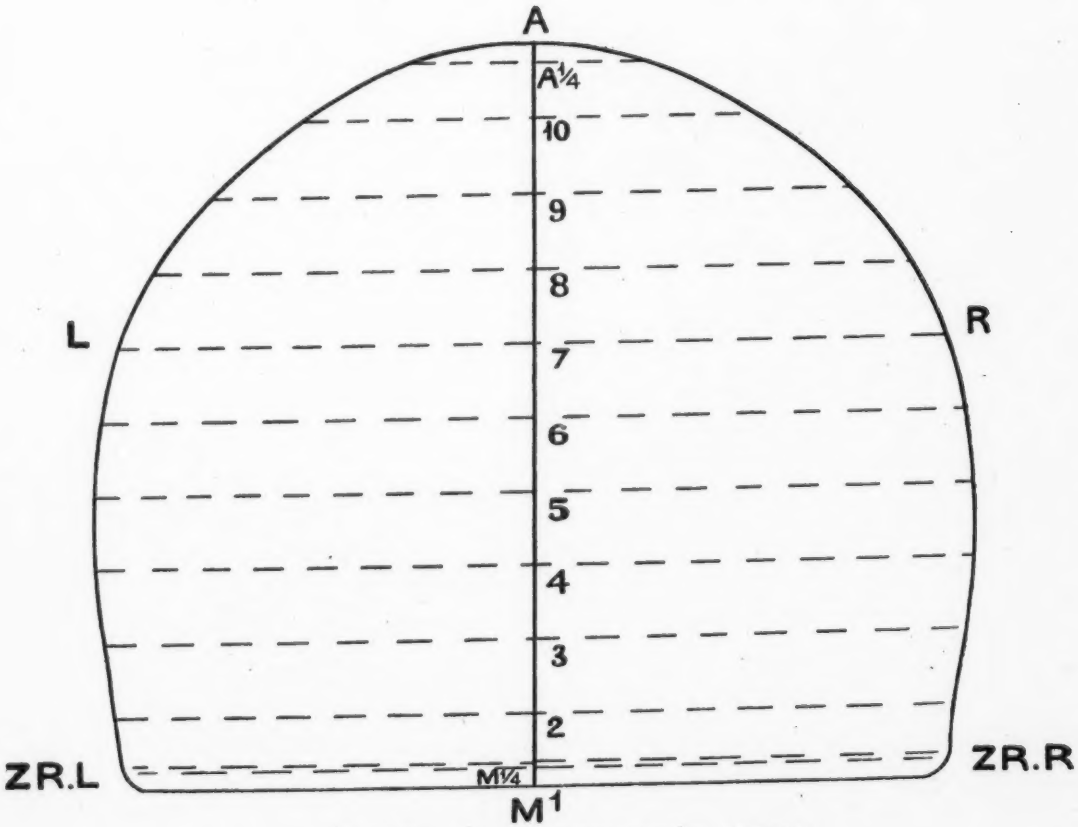

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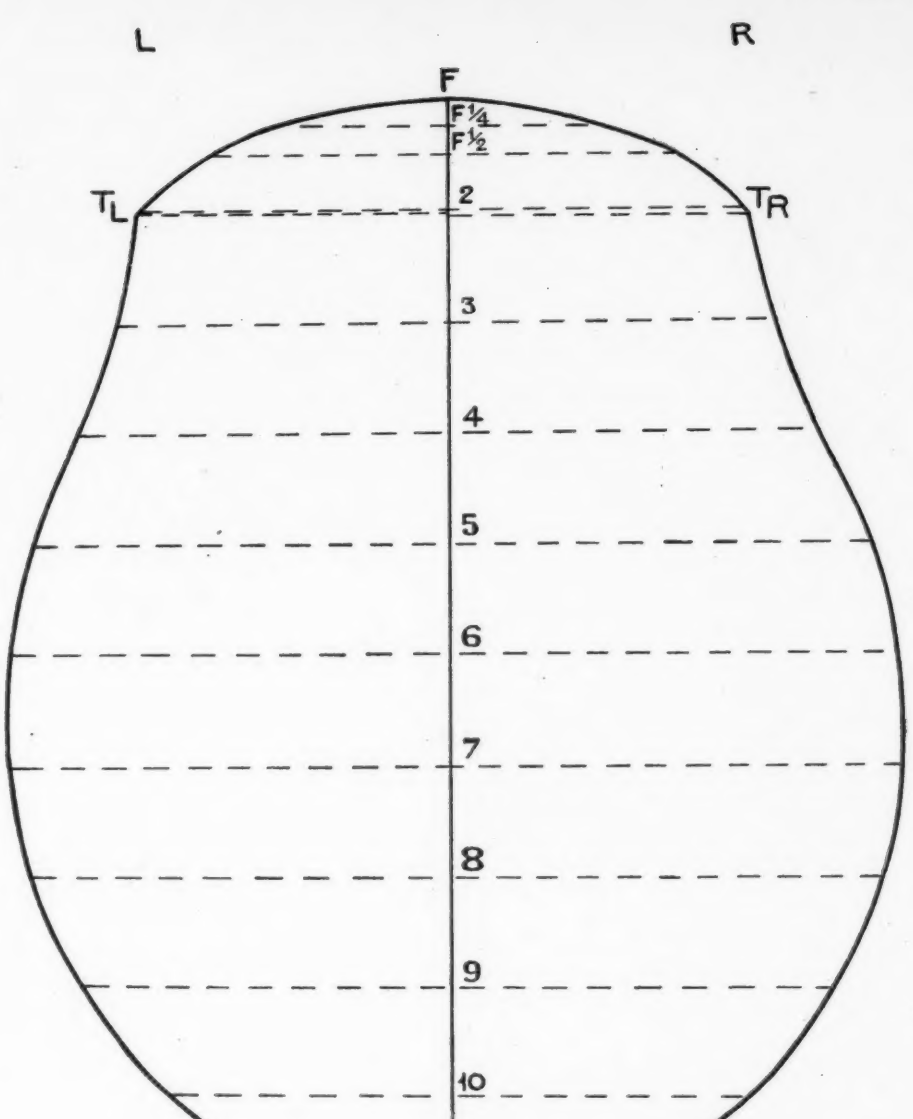

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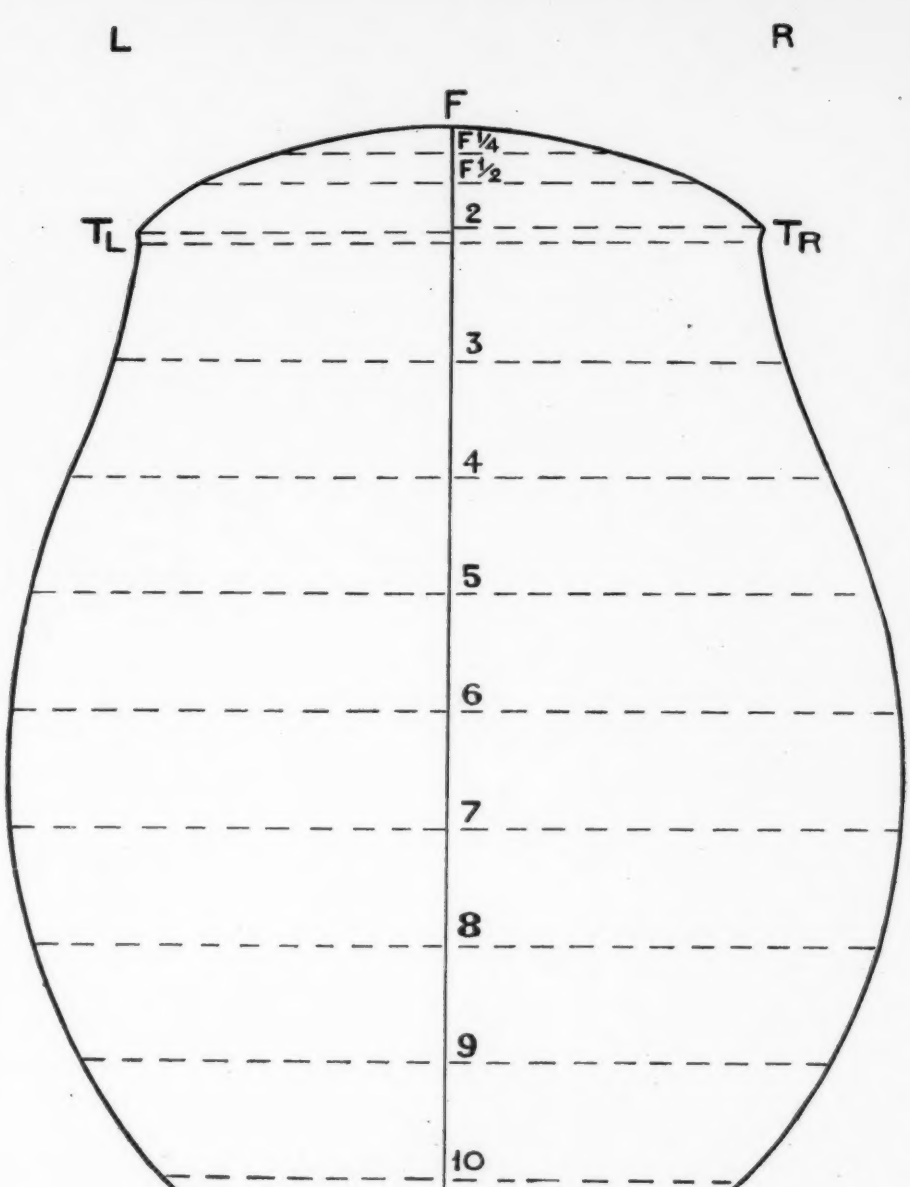

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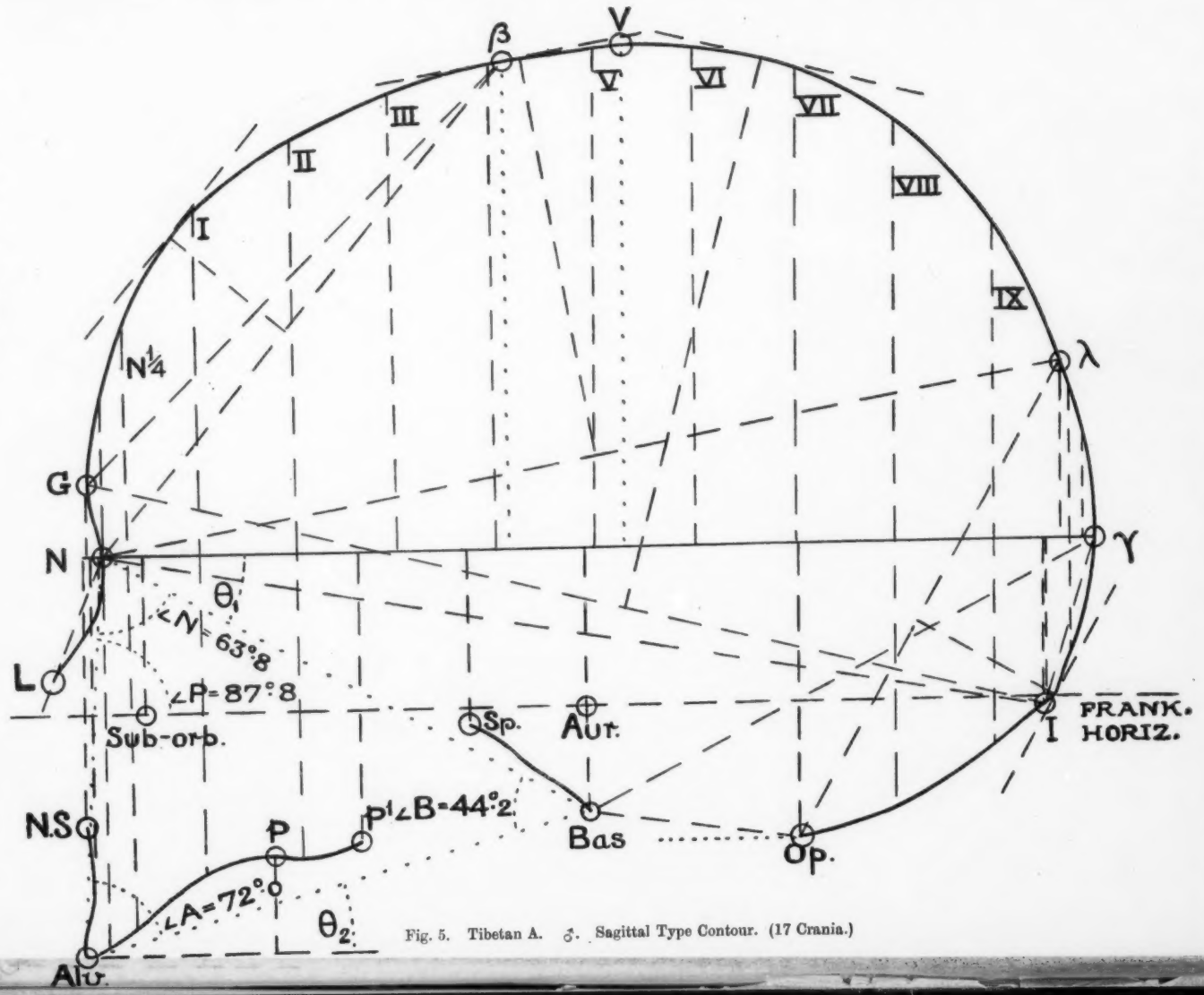

Fig. 5. Tibetan A. ♂. Sagittal Type Contour. (17 Crania.)

Alb.

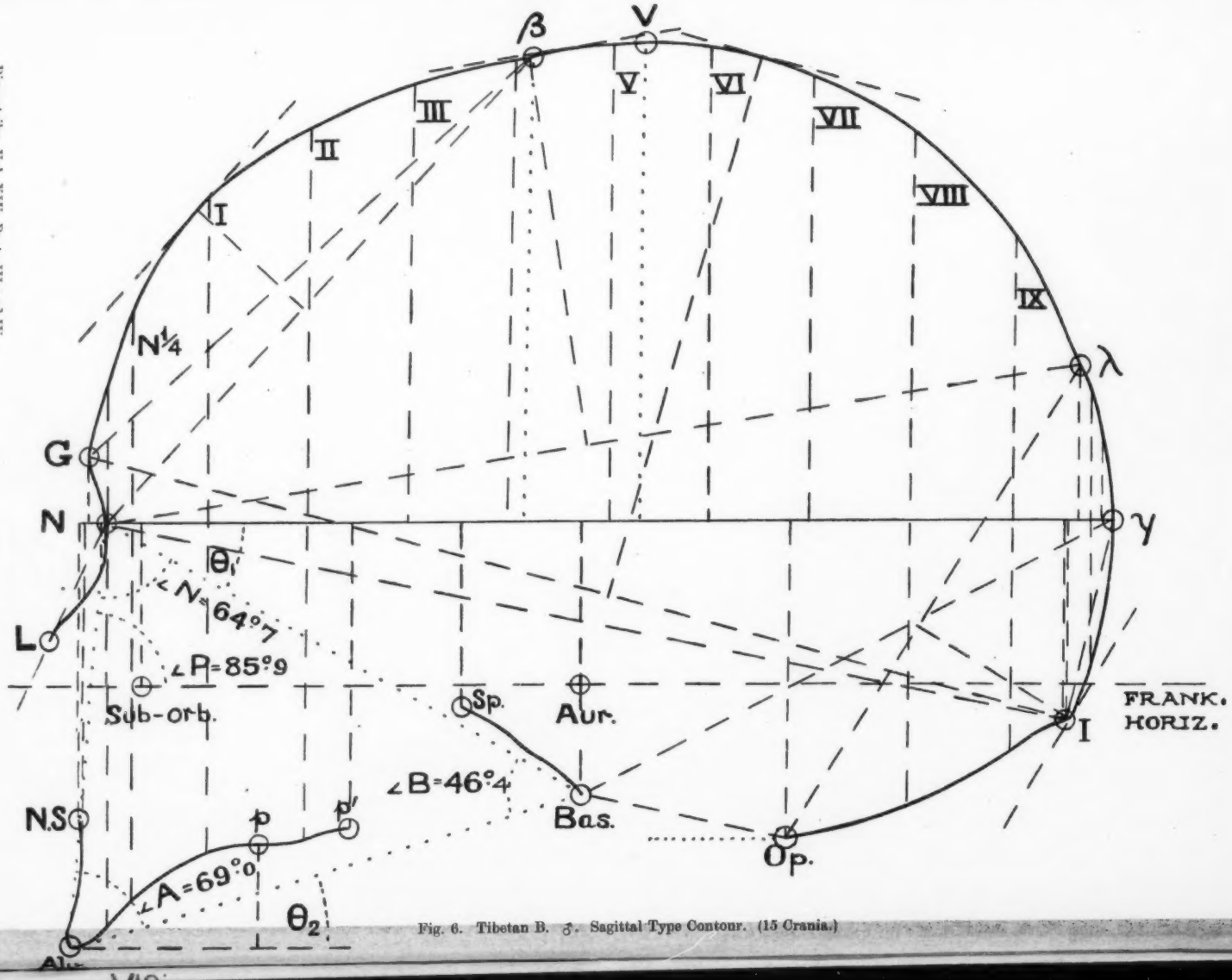

Fig. 6. Tibetan B. ♂. Sagittal Type Contour. (15 Crania.)

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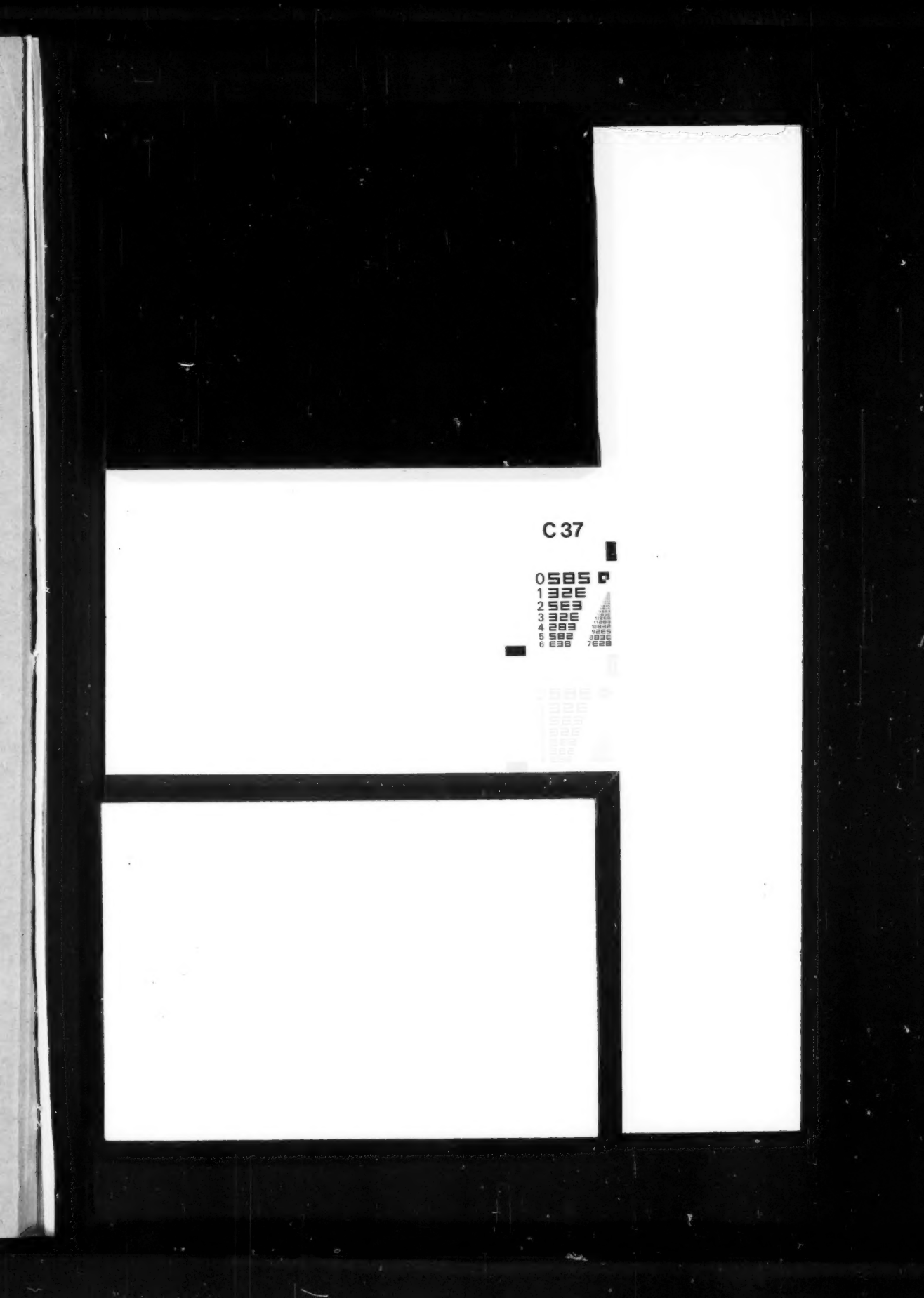
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